Admissibility of Π_2-Inference Rules: interpolation, model completion, and contact algebras

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The symmetric strict implication calculus

Definition

An open subset U of a topological space is called regular open if $U = \text{int}(\text{cl}(U))$.

Let X be a compact Hausdorff space. The set $\text{RO}(X)$ of regular open subsets of X equipped with the well-inside relation $U \prec V$ iff $\text{cl}(U) \subseteq V$ forms a de Vries algebra.

Definition

A de Vries algebra is a complete boolean algebra equipped with a binary relation $\prec$ satisfying

(S1) $0 \prec 0$ and $1 \prec 1$;
(S2) $a \prec b, c$ implies $a \prec b \wedge c$;
(S3) $a, b \prec c$ implies $a \vee b \prec c$;
(S4) $a \leq b \prec c \leq d$ implies $a \prec d$;
(S5) $a \prec b$ implies $a \leq b$;
(S6) $a \prec b$ implies $\neg b \prec \neg a$;
(S7) $a \prec b$ implies there is c with $a \prec c \prec b$;
(S8) $a \neq 0$ implies there is $b \neq 0$ with $b \prec a$.

All the information carried by $(\mathrm{RO}(X), \prec)$ is enough to recover the compact Hausdorff space X up to homeomorphism.

Moreover, every de Vries algebra is isomorphic to one of the form $(\mathrm{RO}(X), \prec)$ for some compact Hausdorff space X.

Theorem (De Vries duality (1962))

The category of compact Hausdorff spaces is dually equivalent to the category of de Vries algebras.

Let $(B, \prec)$ be a de Vries algebra. We can turn $(B, \prec)$ into a boolean algebra with operators by replacing $\prec$ with a binary operator with values in $\{0, 1\}$ (the bottom and top of B).

$$a \rightsquigarrow b = \begin{cases} 1 & \text{if } a \prec b, \\ 0 & \text{otherwise.} \end{cases}$$

$\rightsquigarrow$ is the characteristic function of $\prec \subseteq B \times B$.

Definition

Let $\mathcal{V}$ be the variety generated by de Vries algebras in the language of boolean algebras with a binary operator $\rightsquigarrow$. We call symmetric strict implication algebras the algebras of $\mathcal{V}$.

Definition (G. Bezhanishvili, N. Bezhanishvili, T. Santoli, Y. Venema (2019))

The symmetric strict implication calculus $\mathsf{S^2IC}$ is given by the axioms

- $[\forall]\varphi \leftrightarrow (\top \rightsquigarrow \varphi)$,
- $(\bot \rightsquigarrow \varphi) \wedge (\varphi \rightsquigarrow \top)$,
- $[(\varphi \vee \psi) \rightsquigarrow \chi] \leftrightarrow [(\varphi \rightsquigarrow \chi) \wedge (\psi \rightsquigarrow \chi)]$,
- $[\varphi \rightsquigarrow (\psi \wedge \chi)] \leftrightarrow [(\varphi \rightsquigarrow \psi) \wedge (\varphi \rightsquigarrow \chi)]$,
- $(\varphi \rightsquigarrow \psi) \rightarrow (\varphi \rightarrow \psi)$,
- $(\varphi \rightsquigarrow \psi) \leftrightarrow (\neg\psi \rightsquigarrow \neg\varphi)$,
- $[\forall]\varphi \rightarrow [\forall][\forall]\varphi$,
- $\neg[\forall]\varphi \rightarrow [\forall]\neg[\forall]\varphi$,
- $(\varphi \rightsquigarrow \psi) \leftrightarrow [\forall](\varphi \rightsquigarrow \psi)$,
- $[\forall]\varphi \rightarrow (\neg[\forall]\varphi \rightsquigarrow \bot)$,

and modus ponens (for $\rightarrow$) and necessitation (for $[\forall]$).

Theorem (G. Bezhanishvili, N. Bezhanishvili, T. Santoli, Y. Venema (2019))

$\vdash_{\mathsf{S^2IC}} \varphi$ *iff* $(B, \rightsquigarrow) \vDash \varphi$ *for every symmetric strict impl. algebra* $(B, \rightsquigarrow)$.

$\vdash_{\mathsf{S^2IC}} \varphi$ *iff* $(B, \prec) \vDash \varphi$ *for every de Vries algebra* $(B, \prec)$.

$\vdash_{\mathsf{S^2IC}} \varphi$ *iff* $(\mathrm{RO}(X), \prec) \vDash \varphi$ *for every compact Hausdorff space* X.

Analogous strong completeness results hold.

Therefore, we can think of $\mathsf{S^2IC}$ as the modal calculus of compact Hausdorff spaces where propositional letters are interpreted as regular opens.

When a symmetric strict implication algebra is simple, $\rightsquigarrow$ becomes the characteristic function of a binary relation. Simple symmetric strict implication algebras correspond exactly to contact algebras.

Definition

A contact algebra is a boolean algebra equipped with a binary relation $\prec$ satisfying the axioms:

(S1) $0 \prec 0$ and $1 \prec 1$;
(S2) $a \prec b, c$ implies $a \prec b \wedge c$;
(S3) $a, b \prec c$ implies $a \vee b \prec c$;
(S4) $a \leq b \prec c \leq d$ implies $a \prec d$;
(S5) $a \prec b$ implies $a \leq b$;
(S6) $a \prec b$ implies $\neg b \prec \neg a$.

The variety of symmetric strict implication algebras is a discriminator variety and hence it is generated by its simple algebras which correspond to contact algebras. Therefore,

$$\vdash_{\mathsf{S^2IC}} \varphi \text{ iff } (B, \prec) \vDash \varphi \text{ for every contact algebra } (B, \prec).$$

Therefore, (S7) and (S8) are not expressible in S^2IC.

(S7) $a \prec b$ implies there is c with $a \prec c \prec b$;
(S8) $a \neq 0$ implies there is $b \neq 0$ with $b \prec a$.

What does it mean from the syntactic point of view?

Theorem

For each Π_2-sentence Φ there is an inference rule ρ such that

$$\vdash_{S^2IC+\rho} \varphi \ \text{ iff } \ (B, \prec) \vDash \varphi \text{ for every contact algebra } (B, \prec) \text{ satisfying } \Phi.$$

The rules corresponding to (S7) and (S8) are

$$(\rho_7)\ \frac{(\varphi \rightsquigarrow p) \wedge (p \rightsquigarrow \psi) \to \chi}{(\varphi \rightsquigarrow \psi) \to \chi} \qquad (\rho_8)\ \frac{p \wedge (p \rightsquigarrow \varphi) \to \chi}{\varphi \to \chi}$$

That (S7) and (S8) are not expressible in S^2IC corresponds to the fact that these two rules are admissible in S^2IC.

Π_2-rules

Definition

An inference rule ρ is a Π_2-rule if it is of the form

$$\frac{F(\underline{\varphi}/\underline{x}, \underline{y}) \to \chi}{G(\underline{\varphi}/\underline{x}) \to \chi}$$

where $F(\underline{x}, \underline{y}), G(\underline{x})$ are propositional formulas.

We say that θ is obtained from ψ by an application of the rule ρ if

$$\psi = F(\underline{\varphi}/\underline{x}, \underline{y}) \to \chi \ \text{ and } \ \theta = G(\underline{\varphi}/\underline{x}) \to \chi,$$

where $\underline{\varphi}$ is a tuple of formulas, χ is a formula, and $\underline{y}$ is a tuple of propositional letters not occurring in $\underline{\varphi}$ and χ.

Let $\mathcal{S}$ be a propositional modal system. We say that the rule ρ is admissible in $\mathcal{S}$ if $\vdash_{\mathcal{S}+\rho} \varphi$ implies $\vdash_{\mathcal{S}} \varphi$ for each formula φ.

First method

Conservative extensions

We say that $\varphi(\underline{x}) \wedge \psi(\underline{x}, \underline{y})$ is a conservative extension of $\varphi(\underline{x})$ in $\mathcal{S}$ if

$$\vdash_{\mathcal{S}} \varphi(\underline{x}) \wedge \psi(\underline{x}, \underline{y}) \to \chi(\underline{x}) \ \text{ implies } \ \vdash_{\mathcal{S}} \varphi(\underline{x}) \to \chi(\underline{x})$$

for every formula $\chi(\underline{x})$.

Theorem

If $\mathcal{S}$ has the interpolation property, then a Π_2-rule ρ is admissible in $\mathcal{S}$ iff $G(\underline{x}) \wedge F(\underline{x}, \underline{y})$ is a conservative extension of $G(\underline{x})$ in $\mathcal{S}$.

Therefore, if $\mathcal{S}$ has the interpolation property and conservativity is decidable in $\mathcal{S}$, then Π_2-rules are effectively recognizable in $\mathcal{S}$.

Corollary

The admissibility problem for Π_2-rules is

- coNEXPTIME*-complete in* K *and* S5*;*
- *in* EXPSPACE *and* NEXPTIME*-hard in* S4.

Second method

Uniform interpolants

An S5-modality $[\forall]$ is called a universal modality if

$$\vdash_{\mathcal{S}} \bigwedge_{i=1}^{n} [\forall](\varphi_i \leftrightarrow \psi_i) \to (\Box[\varphi_1, \ldots, \varphi_n] \leftrightarrow \Box[\psi_1, \ldots, \psi_n])$$

for every modality $\Box$ of $\mathcal{S}$.

If $\varphi(\underline{x}, \underline{y})$ is a formula, its right global uniform pre-interpolant $\forall_{\underline{x}}\varphi(\underline{y})$ is a formula such that for every $\psi(\underline{y}, \underline{z})$ we have that

$$\psi(\underline{y}, \underline{z}) \vdash_{\mathcal{S}} \varphi(\underline{x}, \underline{y}) \ \text{ iff } \ \psi(\underline{y}, \underline{z}) \vdash_{\mathcal{S}} \forall_{\underline{x}}\varphi(\underline{y}).$$

Theorem

Suppose that $\mathcal{S}$ has uniform global pre-interpolants and a universal modality $[\forall]$. Then a Π_2-rule ρ is admissible in $\mathcal{S}$ iff

$$\vdash_{\mathcal{S}} [\forall]\forall_{\underline{y}}(F(\underline{x}, \underline{y}) \to z) \to (G(\underline{x}) \to z).$$

Third method

Simple algebras and model completions

To a Π_2-rule we associate the first-order formula

$$\Pi(\rho) := \forall \underline{x}, z \Big(G(\underline{x}) \not\leq z \Rightarrow \exists \underline{y} : F(\underline{x}, \underline{y}) \not\leq z \Big).$$

Theorem

Suppose that $\mathcal{S}$ has a universal modality. A Π_2-rule ρ is admissible in $\mathcal{S}$ iff for each simple $\mathcal{S}$-algebra $\mathcal{B}$ there is a simple $\mathcal{S}$-algebra $\mathcal{C}$ such that $\mathcal{B}$ is a subalgebra of $\mathcal{C}$ and $\mathcal{C} \models \Pi(\rho)$.

In the presence of a universal modality, an $\mathcal{S}$-algebra is simple iff

$$[\forall]x = \begin{cases} 1 & \text{if } x = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, $\mathcal{S}$-algebras form a discriminator variety. Therefore, the variety of $\mathcal{S}$-algebras is generated by the simple $\mathcal{S}$-algebras.

The model completion of a universal first-order theory T, if it exists, is the theory of the existentially closed models of T.

Let T be a universal theory in a finite language. If T is locally finite and has the amalgamation property, then it admits a model completion.

Theorem

Suppose that $\mathcal{S}$ has a universal modality and let $T_{\mathcal{S}}$ be the first-order theory of the simple $\mathcal{S}$-algebras. If $T_{\mathcal{S}}$ has a model completion $T^{\star}_{\mathcal{S}}$, then a Π_2-rule ρ is admissible in $\mathcal{S}$ iff $T^{\star}_{\mathcal{S}} \models \Pi(\rho)$ where

$$\Pi(\rho) := \forall \underline{x}, z \Big(G(\underline{x}) \not\leq z \Rightarrow \exists \underline{y} : F(\underline{x}, \underline{y}) \not\leq z \Big).$$

Model completion of contact algebras and admissibility in S^2IC

Theorem

The theory of contact algebras Con *is locally finite and has the amalgamation property. Therefore, it admits a model completion* $\mathsf{Con}^\star$.

Moreover, the modality $[\forall]$ defined by $[\forall]\varphi := \top \rightsquigarrow \varphi$ is a universal modality. Thus, our third criterion applies.

Proposition

Let $(B, \prec)$ *be a contact algebra. We have that* $(B, \prec)$ *is existentially closed iff for any finite subalgebra* $(B_0, \prec) \subseteq (B, \prec)$ *and for any finite extension* $(C, \prec) \supseteq (B_0, \prec)$ *there exists an embedding* $(C, \prec) \hookrightarrow (B, \prec)$ *such that the following diagram commutes*

$$\begin{array}{ccc} (B, \prec) & & \\ \uparrow & \nwarrow & \\ (B_0, \prec) & \hookrightarrow & (C, \prec) \end{array}$$

Theorem

The model completion $\mathrm{Con}^\star$ *of the theory of contact algebras is finitely axiomatizable.*

An axiomatization is given by the axioms of contact algebras together with the following three sentences.

$$\forall a, b_1, b_2\ (a \neq 0\ \&\ (b_1 \vee b_2) \wedge a = 0\ \&\ a \prec a \vee b_1 \vee b_2 \Rightarrow \exists a_1, a_2\ (a_1 \vee a_2 = a\ \&\ a_1 \wedge a_2 = 0\ \&\ a_1 \neq 0\ \&\ a_2 \neq 0\ \&\ a_1 \prec a_1 \vee b_1\ \&\ a_2 \prec a_2 \vee b_2))$$

$$\forall a, b\ (a \wedge b = 0\ \&\ a \not\prec \neg b \Rightarrow \exists a_1, a_2\ (a_1 \vee a_2 = a\ \&\ a_1 \wedge a_2 = 0\ \&\ a_1 \not\prec \neg b\ \&\ a_2 \not\prec \neg b\ \&\ a_1 \prec \neg a_2))$$

$$\forall a\ (a \neq 0 \Rightarrow \exists a_1, a_2\ (a_1 \vee a_2 = a\ \&\ a_1 \wedge a_2 = 0\ \&\ a_1 \prec a\ \&\ a_1 \not\prec a_1))$$

The two Π_2-rules we saw at the beginning

$$(\rho_7) \quad \frac{(\varphi \rightsquigarrow p) \wedge (p \rightsquigarrow \psi) \to \chi}{(\varphi \rightsquigarrow \psi) \to \chi} \qquad (\rho_8) \quad \frac{p \wedge (p \rightsquigarrow \varphi) \to \chi}{\varphi \to \chi}$$

correspond to the Π_2-sentences

$$\Pi(\rho_7) \quad \forall x_1, x_2, y\,(x_1 \rightsquigarrow x_2 \nleq y \to \exists z : (x_1 \rightsquigarrow z) \wedge (z \rightsquigarrow x_2) \leq y);$$

$$\Pi(\rho_8) \quad \forall x, y\,(x \nleq y \to \exists z : z \wedge (z \rightsquigarrow x) \nleq y).$$

We can use our result to show that these rules are admissible in S^2IC. Indeed, it is sufficient to use the finite axiomatization of $Con^\star$ to show that $Con^\star$ proves $\Pi(\rho_7)$ and $\Pi(\rho_8)$.

The Π_2-rule

$$(\rho_9) \quad \frac{(p \rightsquigarrow p) \wedge (\varphi \rightsquigarrow p) \wedge (p \rightsquigarrow \psi) \to \chi}{(\varphi \rightsquigarrow \psi) \to \chi}$$

corresponds to the Π_2-sentence

$$\Pi(\rho_9) \quad \forall x, y, z\, (x \rightsquigarrow y \not\leq z \to \exists u : (u \rightsquigarrow u) \wedge (x \rightsquigarrow u) \wedge (u \rightsquigarrow y) \not\leq z)$$

which holds in $(\mathsf{RO}(X), \prec)$ iff X is a Stone space.

Using the finite axiomatization it can be shown that $\mathsf{Con}^\star$ proves $\Pi(\rho_9)$. Therefore, we obtain as a corollary that $\mathsf{S^2IC}$ is complete wrt Stone spaces.

Corollary

$\vdash_{\mathsf{S^2IC}} \varphi$ *iff* $\mathsf{RO}(X) \vDash \varphi$ *for every Stone space* X.

This fact was proved in (G. Bezhanishvili, N. Bezhanishvili, T. Santoli, Y. Venema (2019)) using different methods.

THANK YOU!