Existentially Closed Brouwerian Semilattices

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Brouwerian semilattices

Definition

Brouwerian semilattices (also called implicative semilattices) are $\wedge$-semilattices with a top element 1 and an implication operation $\rightarrow$ satisfying

$$a \wedge b \leq c \qquad \text{iff} \qquad a \leq b \rightarrow c$$

Variety of Brouwerian semilattices

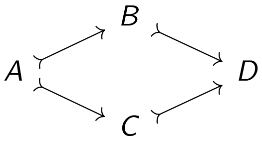
Brouwerian semilattices form an equational class, i.e. a *variety*.

Indeed, Brouwerian semilattices can be equivalently defined as algebras with signature $(\wedge, 1, \to)$ satisfying the following list of equations:

- $a \wedge a = a$
- $a \wedge b = b \wedge a$
- $a \wedge (b \wedge c) = (a \wedge b) \wedge c$
- $a \wedge 1 = a$
- $a \wedge (a \to b) = a \wedge b$
- $b \wedge (a \to b) = b$
- $a \to (b \wedge c) = (a \to b) \wedge (a \to c)$
- $a \to a = 1$

Amalgamation property

The variety of Brouwerian semilattices is amalgamable.
It means that any diagram formed by two embeddings of Brouwerian semilattices having the same domain can be completed to a commutative square entirely made of embeddings.



Interpolation property

Brouwerian semilattices are the algebraic counterpart of the implication-conjunction fragment of intuitionistic logic.

The amalgamation property has a proof-theoretic counterpart regarding such fragment: the *interpolation property*.

For any ϕ, ψ propositional formulas in such fragment there exists a formula θ of the fragment containing only proposition letters common to ϕ and ψ such that $\phi \to \theta$ and $\theta \to \psi$ are validities.

Locally finiteness

The variety is also locally finite: any finitely generated Brouwerian semilattice is actually finite.

Note that this property holds for Boolean algebras but not for Heyting algebras.

The cardinalities of the finitely generated free Brouwerian semilattices, although finite, grow very rapidly.
It is known that

$$\begin{aligned} \#F_0 &= 1 \\ \#F_1 &= 2 \\ \#F_2 &= 18 \\ \#F_3 &= 623,662,965,552,330 \end{aligned}$$

The size of F_4 is still unknown. In [Köh81] it is proved that the number of meet-irreducible elements of F_4 is $2,494,651,862,209,437$.

Existentially closed Brouwerian semilattices

Definition

A Brouwerian semilattice L is said to be existentially closed if for any extension $L \subseteq L'$ and for any existential sentence ϕ in the language of Brouwerian semilattices extended with the names of the elements of L we have that if ϕ is true in L' then it is also true in L.

Existentially closed Brouwerian semilattices

It is a well-known model-theoretic fact that if a universal theory admits a model completion then its existentially closed models are exactly the models of its model completion.

The existence of a model completion of the theory of Brouwerian semilattices is guaranteed by a result due to W. Wheeler [Whe76] because the variety is amalgamable and locally finite. Thus the class of the existentially closed Brouwerian semilattices is elementary.

It is thus natural to look for an axiomatization of said model completion.

Axiomatization

Note that supplying an axiomatization of the model completion for this kind of algebraic theories is usually a hard task. An axiomatization for the model completion of Heyting algebras is still unknown.

Some remarkable exceptions are the case of the locally finite amalgamable varieties of Heyting algebras recently investigated by L. Darnière and M. Junker in [DJ10] and the simpler cases of posets and semilattices studied by M. H. Albert and S. N. Burris in [AB86].
It is well-known that the existentially closed Boolean algebras are exactly the atomless ones.

Axiomatization

We have proven that the following three axioms together with the axioms of Brouwerian semilattices give a finite axiomatization of the model completion of Brouwerian semilattices.

We use the abbreviation $a \ll b$ for $a \leq b$ and $b \to a = a$.

[Density 1] For every c there exists an element b different from 1 such that $b \ll c$.

[Density 2] For every c, a_1, a_2, d such that $a_1, a_2 \neq 1$, $a_1 \ll c$, $a_2 \ll c$ and $d \to a_1 = a_1$, $d \to a_2 = a_2$ there exists an element b different from 1 such that:

$$\begin{aligned} a_1 &\ll b \\ a_2 &\ll b \\ b &\ll c \\ d \to b &= b \end{aligned}$$

Axiomatization

[Splitting] For every a, b_1, b_2 such that $1 \neq a \ll b_1 \wedge b_2$ there exist elements a_1 and a_2 different from 1 such that:

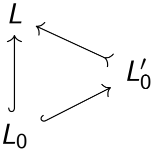
$$b_1 \geq a_1 = a_2 \to a$$
$$b_2 \geq a_2 = a_1 \to a$$
$$a_2 \to b_1 = b_2 \to b_1$$
$$a_1 \to b_2 = b_1 \to b_2$$

Existentially closed Brouwerian semilattices

Thanks to the locally finiteness and the amalgamability, by an easy model-theoretic reasoning we obtain:

Theorem

Let L be a Brouwerian semilattice. L is existentially closed iff for any finite sub-Brouwerian semilattice $L_0 \subseteq L$ and for any finite extension $L'_0 \supseteq L_0$ there exists an embedding $L'_0 \to L$ fixing L_0 pointwise.



We thus want to study finite extensions of finite Brouwerian semilattices.

Finite duality

Any finite Brouwerian semilattice is complete and thus it is a lattice. It is also distributive, because it exists for any element a the right adjoint of $a \wedge -$ given by $a \to -$. Therefore the classes of finite Brouwerian semilattices, of finite bounded distributive lattices and thus also of finite Heyting algebras coincide. Their morphisms, however, do not coincide.

So, we expect the existence of a finite duality between finite Brouwerian semilattices and finite posets.

Finite duality

This is indeed the case. Before giving the full description of the finite duality due to P. Köhler, we shall take a change of perspective.

Instead of working with Brouwerian semilattices we will use the structures obtained by reversing their order. We will call them co-Brouwerian semilattices, CBS for short.

There are two reasons for this: it will make the finite duality easier to work with and it will help to understand intuitively the constructions featured in the proofs.

co-Brouwerian semilattices

Therefore CBS are $\vee$-semilattices with a minimum element 0 and a difference operation with the property

$$a - b \leq c \qquad \text{iff} \qquad a \leq b \vee c$$

Clearly any result concerning Brouwerian semilattices can be translated in the language of CBS by reversing the order: 1 is replaced by 0, meets are replaced by joins and $a \to b$ is replaced by $b - a$.

Finite duality

Theorem

The category of finite CBS is dual to the category **P** *whose objects are finite posets and whose morphisms are partial maps* $f : P \to Q$ *with the following properties:*

- *Strict order preserving: for any* $a, b \in dom\ f$ *if* $a < b$ *then* $f(a) < f(b)$
- *for any* $p \in dom\ f$, $q \in Q$ *if* $f(p) < q$ *then there exists* $p' \in P$ *such that* $p < p'$ *and* $f(p') = q$.

Finite duality

The duality is defined as follows:

To any finite CBS is associated the poset of its join-irreducible elements with the induced order.

On the other hand, to any finite poset P it is associated the CBS $\mathcal{D}(P)$ given by the downward closed subsets of P. The join operation is the set-theoretic union of downsets, the zero element is the empty downset, the difference of two downsets A, B is given by $A - B = \downarrow(A \setminus B)$. Furthermore, to any **P**-morphism between finite posets $f : P \to Q$ it is associated the morphism of CBS $\varphi : \mathcal{D}(Q) \to \mathcal{D}(P)$ given by $\varphi(D) = \downarrow f^{-1}(D)$ for any $D \in \mathcal{D}(Q)$.

Finite duality

It turns out that:

Proposition

Quotients of finite CBS correspond to total and injective **P***-morphisms.*
Embeddings of finite CBS correspond to surjective **P***-morphisms.*

We are in particular interested in the second fact, the one concerning embeddings.

Minimal extensions of finite CBS

Definition

We call an extension of CBS minimal if it cannot be the composition of two proper extensions.

It follows immediately that:

Proposition

Any extension of finite CBS is the composition of a finite chain of minimal extensions.

The following proposition is very useful:

Proposition

The minimal finite extensions of CBS are exactly the ones dual to surjective $\mathbf{P}$*-morphisms* $f : P \to Q$ *such that* $\#P = \#Q + 1$*. (we call these morphisms minimal)*

Minimal surjective **P**-morphisms

There are two kinds of minimal surjective **P**-morphisms:

First kind: the ones in which $\#\text{dom } f = \#P - 1 = \#Q$, i.e. there is exactly one element of P outside the domain and the map is an isomorphism when restricted on its domain.

Second kind: the ones in which f is total, note that in this case all of its fibers contain exactly one element except one fiber having two elements.

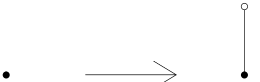
Minimal finite extensions of CBS

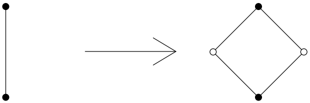
As a result, there are two kinds of minimal finite extensions of finite CBS. The first kind is given by the ones preserving the join-irriducibility of all the elements. The second kind ones preserve the join-irriducibility of all the elements except one which becomes the join of two new join-irreducible elements.

Intuitively, in the first kind ones a new join-irreducible element is added to the domain, whereas in the second kind ones a join-irreducible of the domain is 'splitted' into two new join-irreducibles.

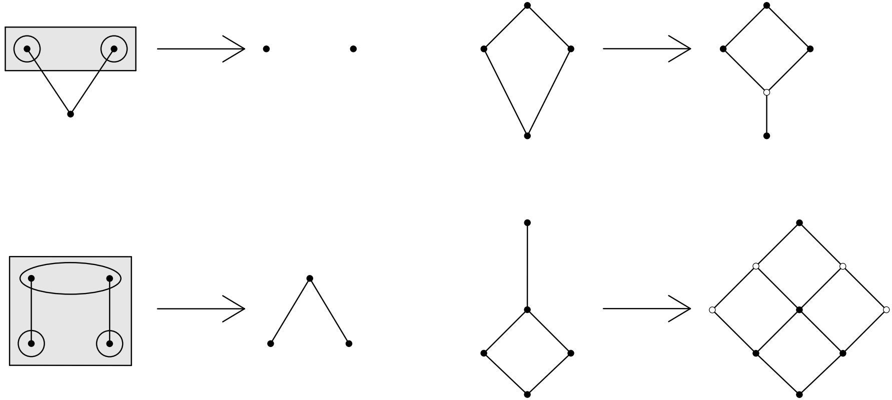
Furthermore any finite minimal extension of the second kind preserves the meets and the top element, i.e. it is a co-Heyting algebras morphism, whereas this is not true for all the minimal finite extensions of the first kind.

Examples





Examples



Minimal finite extensions

Since, as we noted before, any finite extension of CBS is a composition of finite minimal ones, we can replace finite extensions with minimal finite extensions in the characterization of the existentially closed structures we have given before.

As a consequence of this, we are interested in studying all the possible minimal finite extensions of any finite CBS.

Footprints of minimal finite extensions

It turns out that, given a finite poset Q, the minimal surjective **P**-morphisms with codomain Q correspond up to isomorphism to some 'footprints' inside Q.

The ones of the first kind correspond to couples (D, U) where D, U are respectively a downset and an upset of Q such that $D \cap U = \emptyset$ and for any $d \in D$ and $u \in U$ we have $d \leq u$. Intuitively, D and U tell us where the new join-irreducible element is going to be.

The ones of the second kind correspond to triples (g, D_1, D_2) where $g \in Q$ and D_1, D_2 are downsets of Q such that $D_1 \cup D_2 = \downarrow g \setminus \{g\}$. Intuitively, D_1, D_2 tell us how g is 'splitted' in two.

Minimal finite extensions

Theorem

Any extension of a finite CBS L_0 is finite and minimal iff is generated over L_0 by either an element x such that:

- $x \notin L_0$

and for any a join-irreducible of L_0:

- $a - x \in L_0$
- $x - a = x$ *or* $x - a = 0$

or by two elements x_1, x_2 such that:

- $x_1, x_2 \notin L_0$ *and* $x_1 \neq x_2$

and there exists g join-irreducible element of L_0 such that:

- $g - x_1 = x_2$ *and* $g - x_2 = x_1$
- *For any a join-irreducible of L_0 such that $a < g$ we have $a - x_i \in L_0$ for $i = 1, 2$*

Existentially closed CBS

In this way, we can obtain a characterization of the existentially closed CBS: a CBS L is existentially closed iff for any finite sub-CBS $L_0 \subseteq L$ and for any 'footprint' inside L_0 there is minimal finite extension of L_0 inside L having that 'footprint'.

This characterization is stated precisely by the following theorem.

Existentially closed CBS

Theorem

A CBS L is existentially closed iff it satisfies the two following conditions:
1) *For any finite sub-CBS $L_0 \subseteq L$, for any $g_1, \ldots, g_k$ $(k \geq 0)$ join-irreducibles of L_0 and $h \in L_0$ such that $h < g_i$ for $i = 1, \ldots, k$, there exists $x \in L \setminus L_0$ such that for any a join-irreducible of L_0:*

- $a - x \in L_0$.
- $x - a = x$ *or* $x - a = 0$.
- $a < x$ *iff* $a \leq h$ *and* $x < a$ *iff* $g_i \leq a$ *for some* $i = 1, \ldots, k$.

2) *For any finite sub-CBS $L_0 \subseteq L$, for any g join-irreducible of L_0 and $h_1, h_2 \in L_0$ such that $h_1 \vee h_2$ is the unique predecessor of g in L_0, there exist $x_1, x_2 \in L \setminus L_0$ such that $x_1 \neq x_2$ and for any a join-irreducible of L_0:*

- $g - x_1 = x_2$ *and* $g - x_2 = x_1$.
- *if* $a < g$ *then* $a - x_i \in L_0$ *for* $i = 1, 2$.
- $a < x_i$ *iff* $a \leq h_i$ *for* $i = 1, 2$.

Existentially closed CBS

These two conditions can be written as two infinite schemata of first-order axioms.

But this axiomatization is clearly unsatisfactory: other than consisting of an infinite number of axioms, the conditions it expresses are really convoluted and hard to work with.

Nonetheless, it will come in handy when we will want to prove that any CBS satisfying the Density 1, 2 and Splitting axioms is existentially closed, it is sufficient to prove that it satisfies the conditions above.

Density axioms

$a \ll b$ means $a \leq b$ and $b - a = b$.

[Density 1 Axiom] For every c there exists $b \neq 0$ such that $c \ll b$

[Density 2 Axiom] For every c, a_1, a_2, d such that $a_1, a_2 \neq 0$, $c \ll a_1$, $c \ll a_2$ and $a_1 - d = a_1$, $a_2 - d = a_2$ there exists an element b different from 0 such that:

$$\begin{aligned} c &\ll b \\ b &\ll a_1 \\ b &\ll a_2 \\ b - d &= b \end{aligned}$$

Splitting axiom

[Splitting Axiom] For every a, b_1, b_2 such that $b_1 \vee b_2 \ll a \neq 0$ there exist elements a_1 and a_2 different from 0 such that:

$$
\begin{aligned}
a - a_1 &= a_2 \geq b_2 \\
a - a_2 &= a_1 \geq b_1 \\
b_2 - a_1 &= b_2 - b_1 \\
b_1 - a_2 &= b_1 - b_2
\end{aligned}
$$

Axiomatization

To prove the validity of these axioms on all the existentially closed CBS we can use the following result which follows easily from the characterization of existentially closed CBS given before about the finite extensions of finite sub-CBS.

Lemma

Let $\theta(\underline{x})$ and $\phi(\underline{x}, \underline{y})$ be quantifier-free formulas in the language of CBS. Assume that for every finite CBS L_0 and every tuple $\underline{a}$ of elements of L_0 such that $L_0 \vDash \theta(\underline{a})$, there exists an extension L_1 of L_0 which satisfies $\exists \underline{y} \phi(\underline{a}, \underline{y})$.
Then every existentially closed CBS satisfies the following sentence:

$$\forall \underline{x}(\theta(\underline{x}) \longrightarrow \exists \underline{y} \phi(\underline{x}, \underline{y}))$$

In our cases, to construct the required extension L_1 is relatively easy using the finite duality.

Axiomatization

Proving that any CBS satisfying these axioms is existentially closed is the hardest part.

We sketch the idea of the first part of this proof.

We want to prove that given a CBS L satisfying the Splitting axiom, $L_0 \subseteq L$ a finite sub-CBS, g join-irreducible element of L_0 and $h_1, h_2 \in L_0$ such that $h_1 \vee h_2 \ll g$ there exists $L_0' \subseteq L$ such that $L_0 \subseteq L_0'$ is a minimal extension of the second kind of L_0 corresponding to the 'footprint' (g, h_1, h_2).

Axiomatization

The proof is by induction on a natural number n that is associated to (h_1, h_2) which measures how much smaller is $h_1 \wedge h_2$ (taken inside L_0) than h_1 and h_2.

The idea is to use the Splitting axiom to split g in two parts, one over h_1 and another over h_2. Then splitting again each of these two parts and so on, we stop after at most n iterations of this process. We group all these elements we have obtained by splitting into two sets and we take the joins of these two sets.

In this way we obtain two elements of L that generate over L_0 precisely an extension L_0' which is a minimal extension of the second kind corresponding to the 'footprint' (g, h_1, h_2).

Some properties of ex. closed Brouwerian semilattices

Finally, we present some properties of the existentially closed Brouwerian semilattices which can be deduced easily from our investigation.

In any existentially closed Brouwerian semilattice:

- there is no bottom element
- the join of any pair of incomparable elements doesn't exist
- there are no meet-irreducible elements

Thanks for your attention!

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