Baker-Beynon and Marra-Spada dualities beyond semisimplicity

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Baker-Beynon and Marra-Spada

Theorem (Baker-Beynon duality)

- *The category of semisimple abelian ℓ-groups is dually equivalent to the category of closed cones in $\mathbb{R}^\kappa$ and piecewise homogeneous linear maps with integer coefficients.*
- *The category of semisimple vector lattices is dually equivalent to the category of closed cones in $\mathbb{R}^\kappa$ and piecewise homogeneous linear maps with real coefficients.*

Baker-Beynon and Marra-Spada

Theorem (Marra-Spada duality)

- *The category of semisimple MV-algebras is dually equivalent to the category of closed subsets of* $[0,1]^\kappa$ *and piecewise linear maps with integer coefficients.*
- *The category of semisimple Riesz MV-algebras is dually equivalent to the category of closed subsets of* $[0,1]^\kappa$ *and piecewise linear maps with real coefficients.*

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- vector lattices,
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- Riesz MV-algebras.

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In the following, V will denote either one of the varieties of:

- abelian ℓ-groups, (ℓ-ideals = convex subgroups)
- vector lattices, (ℓ-ideals = convex vector subspaces)
- MV-algebras, (ideals = lattice ideals closed under $\oplus$)
- Riesz MV-algebras. (ideals = lattice ideals closed under $\oplus$ and scalar multiplication)

Congruences in these varieties correspond to ideals.

Basic Galois connection

Fix $A \in V$.

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Let κ be a cardinal, and $\mathscr{F}_\kappa$ the free algebra in V over κ generators.

For any $T \subseteq \mathscr{F}_\kappa$ and $S \subseteq A^\kappa$, we define the following operators.

$$\begin{aligned} \mathbb{V}_A(T) =& \{x \in A^\kappa \mid t(x) = 0 \text{ for all } t \in T\} \\ \mathbb{I}_A(S) =& \{t \in \mathscr{F}_\kappa \mid t(x) = 0 \text{ for all } x \in S\}. \end{aligned}$$

$\mathbb{I}_A(S)$ is always an ideal.

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$$T \subseteq \mathbb{I}_A(S) \quad \text{iff} \quad S \subseteq \mathbb{V}_A(T)\ .$$

Fixpoints of the Galois connection

Algebraic Nullstellensatz (Caramello, Marra, and Spada 2021)

- Let I be an ideal of $\mathscr{F}_\kappa$. We have $I = \mathbb{I}_A(x)$ for some $x \in A^\kappa$ iff $\mathscr{F}_\kappa / I$ embeds into A.

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Definition

The subsets $\mathbb{V}_A(I) = \{x \in A^\kappa \mid t(x) = 0 \text{ for all } t \in I\}$ are the closed subsets of a topology on A^κ called the Zariski topology.

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The fixpoints of the Galois connection are:

- the intersections of ideals I of $\mathscr{F}_\kappa$ such that $\mathscr{F}_\kappa / I$ embeds into A,
- the Zariski closed subsets of A^κ.

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Theorem (Caramello, Marra, and Spada 2021)

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- *the category of algebras of V that are subdirect products of subalgebras of A, and*
- *the category of Zariski closed subsets C of A^κ where κ ranges over all the cardinal numbers.*

$$\mathscr{F}_\kappa / I \longrightarrow \mathbb{V}_A(I)$$

$$\mathscr{F}_\kappa / \mathbb{I}_A(C) \longleftarrow C$$

Baker-Beynon duality

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We obtain the Baker-Beynon duality by applying the general affine duality approach with $A = \mathbb{R}$.

Applying the general affine duality approach with $A = \mathbb{R}$

Theorem

An abelian ℓ-group embeds into $\mathbb{R}$ iff it is simple or trivial. Moreover, every simple vector lattice is isomorphic to $\mathbb{R}$.

Thus, semisimple abelian ℓ-group/vector lattices are exactly the subdirect products of subalgebras of $\mathbb{R}$.

Definition

A subset of $\mathbb{R}^\kappa$ is a cone if it is closed under multiplication by nonnegative scalars.

The Zariski closed subsets of $\mathbb{R}^\kappa$ are the cones that are closed subsets wrt the euclidean topology.

It remains to describe the functors.

Piecewise linear functions

Definition

A continuous function $f : \mathbb{R}^\kappa \to \mathbb{R}$ is piecewise linear if there exist $g_1, \ldots, g_n$ linear polynomials in the variables $(x_\alpha)_{\alpha<\kappa}$ such that for each $x \in \mathbb{R}^\kappa$ we have $f(x) = g_i(x)$ for some $i = 1, \ldots, n$.

- The set $\mathrm{PWL}_{\mathbb{R}}(\mathbb{R}^\kappa)$ of piecewise linear homogeneous functions on $\mathbb{R}^\kappa$ is a vector lattice with pointwise operations.

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- The set $\mathrm{PWL}_\mathbb{Z}(\mathbb{R}^\kappa)$ of piecewise linear homogeneous functions on $\mathbb{R}^\kappa$ such that $g_1, \ldots, g_n$ have integer coefficients is an abelian ℓ-group with pointwise operations.

Theorem

- $\mathrm{PWL}_\mathbb{R}(\mathbb{R}^\kappa)$ *is iso to the free vector lattice on* κ *generators.*
- $\mathrm{PWL}_\mathbb{Z}(\mathbb{R}^\kappa)$ *is iso to the free abelian* ℓ*-group on* κ *generators.*

The PWL functors

If $X \subseteq \mathbb{R}^\kappa$, we denote by $\mathrm{PWL}_\mathbb{R}(X)$ and $\mathrm{PWL}_\mathbb{Z}(X)$ the sets of piecewise linear homogeneous maps restricted to X.

- $\mathscr{F}_\kappa \,/\, \mathbb{I}_\mathbb{R}(C) \cong \mathrm{PWL}_\mathbb{R}(C)$ (vector lattices)
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Theorem (Baker 1968)

- *Every κ-generated semisimple vector lattice is isomorphic to* $\mathrm{PWL}_\mathbb{R}(C)$ *where C is a cone that is closed in $\mathbb{R}^\kappa$.*
- *Every κ-generated semisimple abelian ℓ-group is isomorphic to* $\mathrm{PWL}_\mathbb{Z}(C)$ *where C is a cone that is closed in $\mathbb{R}^\kappa$.*

The $\mathbb{V}_{\mathbb{R}}$ functors

Any semisimple abelian ℓ-group/vector lattice A can be represented as a quotient $\mathscr{F}_{\kappa}/I$.

To A we associate the closed cone $\mathbb{V}_{\mathbb{R}}(I)$ of $\mathbb{R}^{\kappa}$.

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Baker-Beynon duality

Theorem (Beynon 1974)

- *The category of finitely generated archimedean vector lattices is dually equivalent to the category of closed cones in $\mathbb{R}^n$ for $n \in \mathbb{N}$ and piecewise linear homogeneous maps with real coefficients.*
- *The category of finitely generated archimedean abelian ℓ-groups is dually equivalent to the category of closed cones in $\mathbb{R}^n$ for $n \in \mathbb{N}$ and piecewise linear homogeneous maps with integer coefficients.*

Beyond Baker-Beynon duality

Looking for the right A

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- A/I is linearly ordered iff I is prime.
- Every ℓ-ideal is intersection of prime ℓ-ideals.
- Every abelian ℓ-group/vector lattice is subdirect product of linearly ordered ones.

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To apply the general affine duality approach we need A such that every linearly ordered abelian ℓ-group/vector lattice embeds into A.

This is not possible for cardinality reasons. However, such an A exists if we impose a bound on the cardinality/number of generators.

The ultrapower $\mathcal{U}$

Theorem

Let γ be a cardinal. There exists an ultrapower $\mathcal{U}$ of $\mathbb{R}$ such that every κ-generated linearly ordered abelian ℓ-group/vector lattice with $\kappa \leq \gamma$ embeds into $\mathcal{U}$.

- Any linearly ordered abelian ℓ-group G can be embedded into a divisible linearly ordered abelian ℓ-group $D(G)$.
- Any linearly ordered divisible abelian ℓ-group is elementary equivalent to $\mathbb{R}$.
- $D(G)$ embeds into an ultrapower $\mathcal{U}$ of $\mathbb{R}$.
- $\mathcal{U}$ only depends on the cardinality of G.

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Theorem (C., Lapenta, and Spada)

Let γ be a cardinal. There exists an ultrapower $\mathcal{U}$ of $\mathbb{R}$ such that:

- *The category of κ-generated vector lattices for some $\kappa \leq \gamma$ is dually equivalent to the category of Zariski closed subsets of $\mathcal{U}^\kappa$ for some $\kappa \leq \gamma$.*

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The Zariski topology on $\mathcal{U}^\kappa$ depends on whether we work with abelian ℓ-groups or vector lattices.

Enlargements of piecewise linear functions

Every piecewise linear function $f : \mathbb{R} \to \mathbb{R}$ can be extended to a function ${}^*f : \mathcal{U} \to \mathcal{U}$ by setting ${}^*f([(r_i)_{i \in I}]) = [(f(r_i))_{i \in I}]$.

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We define:

${}^*\mathrm{PWL}_\mathbb{R}(\mathcal{U}^\kappa) = \{{}^*f \mid f \in \mathrm{PWL}_\mathbb{R}(\mathbb{R}^\kappa)\}$,

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If $X \subseteq \mathcal{U}^\kappa$, we can consider ${}^*\mathrm{PWL}_{\mathbb{R}}(X)$ and ${}^*\mathrm{PWL}_{\mathbb{Z}}(X)$.

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Proposition

Let C be a Zariski closed subset of $\mathcal{U}^\kappa$.

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Theorem

- *Every κ-generated vector lattice is isomorphic to ${}^{*}\mathrm{PWL}_{\mathbb{R}}(C)$ where C is a Zariski closed of $\mathcal{U}^{\kappa}$.*
- *Every κ-generated abelian ℓ-group is isomorphic to ${}^{*}\mathrm{PWL}_{\mathbb{Z}}(C)$ where C is a Zariski closed of $\mathcal{U}^{\kappa}$.*

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The Zariski topology on $\mathcal{U}^n$

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Irreducible closed in $\mathcal{U}^n$ are exactly the closure of points. They are the subsets $\mathbb{V}_{\mathcal{U}}(I)$ with I prime or $I = \mathscr{F}_n$.

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The irreducible Zariski-closed subsets of $\mathbb{R}^n$ are the semilines starting from the origin ($\mathbb{V}_{\mathbb{R}}(I)$ with I maximal) and the origin ($\mathbb{V}_{\mathbb{R}}(I)$ with $I = \mathscr{F}_n$).

Indices and irreducible closed

Orthogonal decomposition theorem (Goze 1995)

If $x \in \mathcal{U}^n$, then x can be written in a unique way as $\alpha_1 v_1 + \cdots + \alpha_k v_k$ with $v_1, \ldots, v_k$ orthonormal vectors of $\mathbb{R}^n$ and $0 < \alpha_1, \ldots, \alpha_k \in \mathcal{U}$ such that α_{i+1}/α_i is infinitesimal.

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Thus, we can associate to each $x \in \mathcal{U}^n$ the sequence $\mathbf{v} = (v_1, \ldots, v_k)$ of orthonormal vectors.

We call such sequences indices.

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Theorem (C., Lapenta, Spada)

In the Zariski topology of $\mathcal{U}^n$ relative to vector lattices each irreducible closed of $\mathcal{U}^n$ is $\mathrm{Cone}(\mathbf{v})$ *for some index* $\mathbf{v}$.

Indices and cones

Every subset $X \subseteq \mathbb{R}^n$ can be associated with a subset ${}^*X$ of $\mathcal{U}^n$ called the enlargement of X. Every predicate $P \subseteq \mathbb{R}^n$ and function $f : \mathbb{R}^n \to \mathbb{R}$ can be enlarged to ${}^*P \subseteq \mathcal{U}^n$ and ${}^*f : \mathcal{U}^n \to \mathcal{U}$.

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Transfer principle (Łoś Theorem)

Let φ be a first order formula and ${}^*\varphi$ the formula obtained by replacing every predicate symbol P and every function symbol f with ${}^*P$ and ${}^*f$. Then φ is true in $\mathbb{R}$ iff ${}^*\varphi$ is true in $\mathcal{U}$.

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If $\mathbf{v}$ is an index, we say that a closed cone of $\mathbb{R}^n$ is a $\mathbf{v}$-cone if there exist real numbers $r_2, \ldots, r_k > 0$ such that the cone is generated by $\{v_1, v_1 + r_2 v_2, \ldots, v_1 + r_2 v_2 + \cdots + r_k v_k\}$.

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Transfer principle (Łoś Theorem)

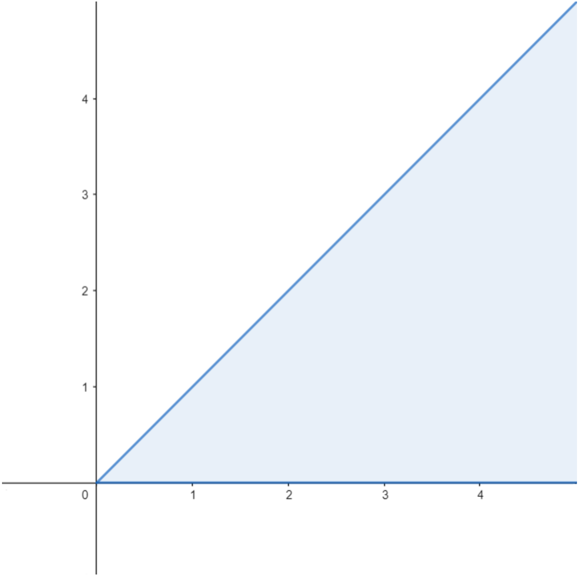
Let φ be a first order formula and ${}^*\varphi$ the formula obtained by replacing every predicate symbol P and every function symbol f with ${}^*P$ and ${}^*f$. Then φ is true in $\mathbb{R}$ iff ${}^*\varphi$ is true in $\mathcal{U}$.

If $\mathbf{v}$ is an index, we say that a closed cone of $\mathbb{R}^n$ is a **$\mathbf{v}$**-cone if there exist real numbers $r_2, \ldots, r_k > 0$ such that the cone is generated by $\{v_1, v_1 + r_2 v_2, \ldots, v_1 + r_2 v_2 + \cdots + r_k v_k\}$.

Proposition

Cone($\mathbf{v}$) is the intersection of the enlargements of all the $\mathbf{v}$-cones.

$\mathbf{v} = ((1, 0), (0, 1)).$



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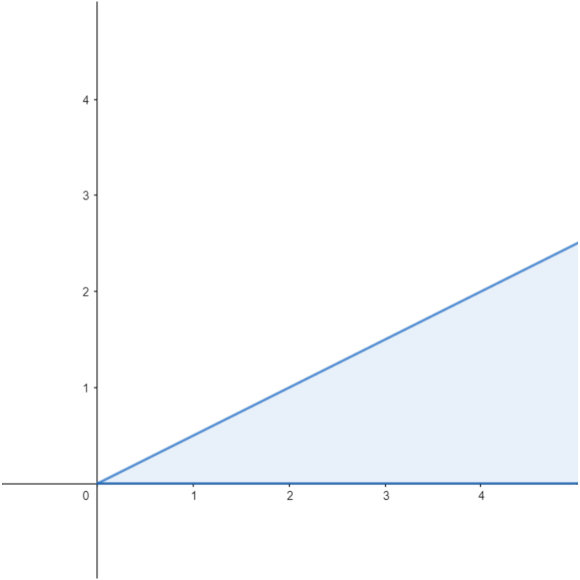

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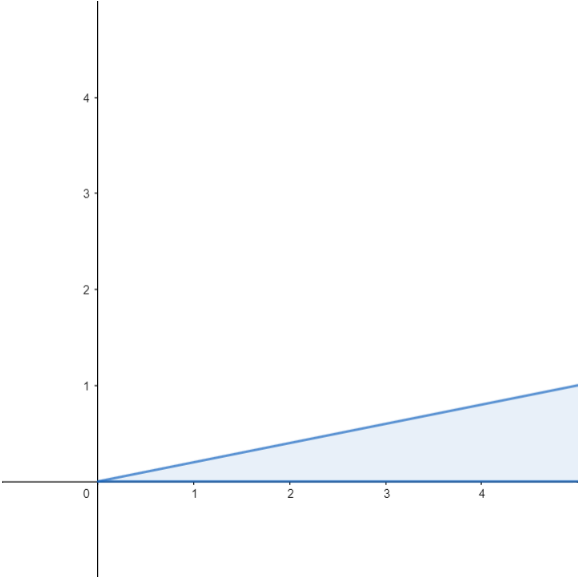

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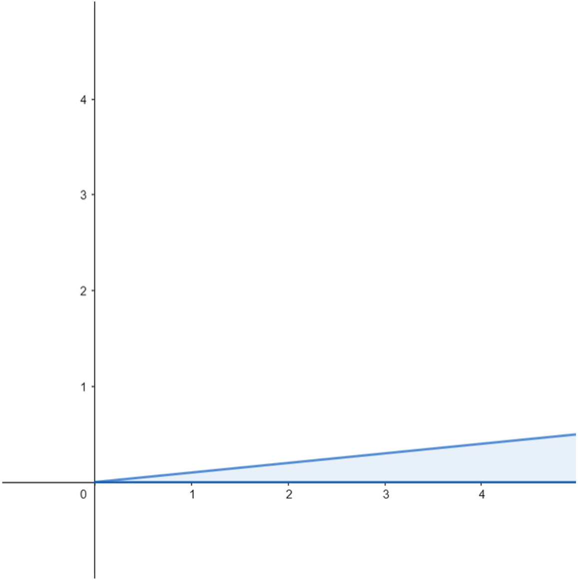

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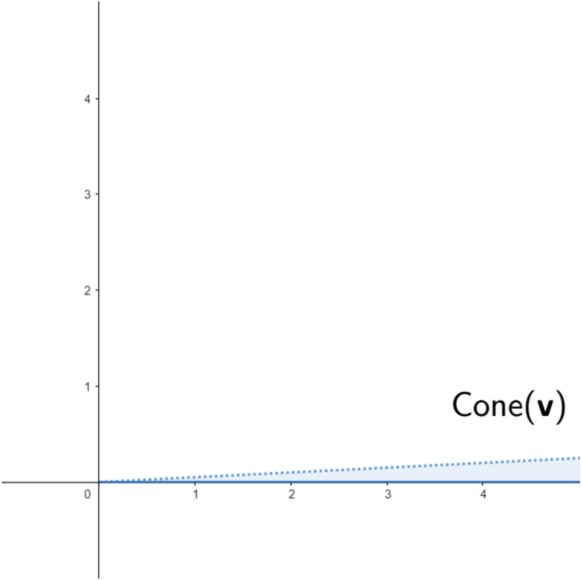

Primes and indices

Theorem (C., Lapenta, and Spada)

If $f \in \mathrm{PWL}_{\mathbb{R}}(\mathbb{R}^n)$*, then* ${}^*f$ *vanishes on* $\mathrm{Cone}(\mathbf{v})$ *iff* f *vanishes on some* $\mathbf{v}$*-cone.*

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Theorem (Panti 1999)

Each prime ℓ-ideal of the vector lattice $\mathscr{F}_n$ is of the form $\{f \in \mathrm{PWL}_{\mathbb{R}}(\mathbb{R}^n) \mid f \text{ vanishes on a } \mathbf{v}\text{-cone}\}$ for some index $\mathbf{v}$.

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Fix a positive infinitesimal $\varepsilon \in \mathcal{U}$. If I is the prime ℓ-ideal of the vector lattice $\mathscr{F}_n$ associated with the index $\mathbf{v} = (v_1, \ldots, v_k)$, then $\mathbb{V}_{\mathcal{U}}(I) = \mathrm{cl}\{v_1 + \varepsilon v_2 + \cdots + \varepsilon^{k-1} v_k\}$.

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This allows to embed the spectrum of a finitely generated vector lattice V into its dual cone so that $V \cong {}^*\mathrm{PWL}_{\mathbb{R}}(\mathtt{Spec}(V))$.

Abelian ℓ-groups and $\mathbb{Z}$-reduced indices

Definition

If $w \in \mathbb{R}^n$, let $\langle w \rangle$ be the smallest subspace containing w that admits a basis in $\mathbb{Z}^n$.

An index $\mathbf{v} = (v_1, \ldots, v_k)$ is $\mathbb{Z}$-reduced if $\langle v_i \rangle$ and $\langle v_j \rangle$ are orthogonal for each $i \neq j$.

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Theorem (C., Lapenta, and Spada)

In the Zariski topology of $\mathcal{U}^n$ relative to abelian ℓ-groups each irreducible closed of $\mathcal{U}^n$ is of the form

$$\bigcup \{\mathrm{Cone}(\mathbf{w}) \mid \mathrm{red}(\mathbf{w}) = \mathbf{v}\}.$$

for some $\mathbb{Z}$-reduced index $\mathbf{v}$.

Marra-Spada duality and beyond

Marra-Spada duality

Let V be the variety of MV-algebras/Riesz MV-algebras. If we take $A = [0, 1]$, the general affine duality approach yields the Marra-Spada duality.

Theorem (Marra, Spada 2012)

- *The category of semisimple MV-algebras is dually equivalent to the category of closed subsets of $[0, 1]^\kappa$ and piecewise linear maps with integer coefficients.*
- *The category of semisimple Riesz MV-algebras is dually equivalent to the category of closed subsets of $[0, 1]^\kappa$ and piecewise linear maps with real coefficients.*

Theorem (C., Lapenta, and Spada)

Let γ be a cardinal. There exists an ultrapower $\mathcal{U}$ of $[0, 1]$ such that:

- *The category of κ-generated MV-algebras for some $\kappa \leq \gamma$ is dually equivalent to the category of Zariski closed subsets of $\mathcal{U}^\kappa$ for some $\kappa \leq \gamma$.*
- *The category of κ-generated Riesz MV-algebras for some $\kappa \leq \gamma$ is dually equivalent to the category of Zariski closed subsets of $\mathcal{U}^\kappa$ for some $\kappa \leq \gamma$.*

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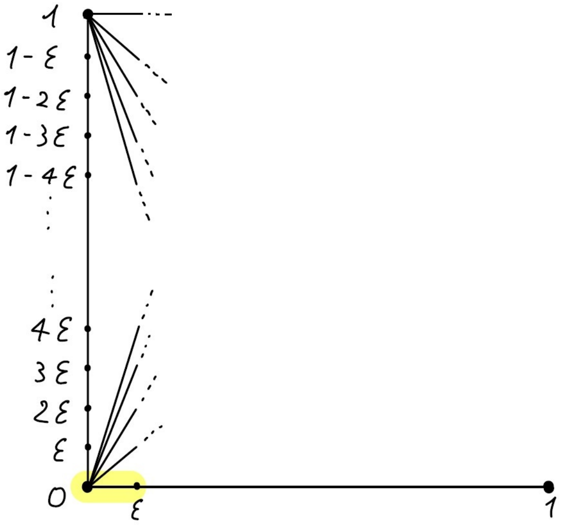
- The prime ideal corresponding to the index $(x, v_1, \dots, v_k)$ is given by the piecewise linear maps that vanish on a $(v_1, \dots, v_n)$-simplex with vertex x.

Example: Chang



THANK YOU!

Abelian ℓ-groups and vector lattices

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Abelian ℓ-groups and vector lattices form varieties.

ℓ-ideals

Congruences in abelian ℓ-groups and vector lattices correspond to ℓ-ideals.

Definition

- An ℓ-ideal in an abelian ℓ-group is a subgroup I that is convex, i.e. $|a| \leq |b|$ and $b \in I$ imply $a \in I$.
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Definition

- A proper ℓ-ideal is called maximal if it is maximal wrt inclusion.
- A nontrivial abelian ℓ-group/vector lattice A is simple if $\{0\}$ and A are the only ℓ-ideals of A.

Archimedeanity

Definition

An abelian ℓ-group/vector lattice is semisimple if the intersection of all its maximal ℓ-ideals is $\{0\}$.

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Semisimple $\Rightarrow$ archimedean
Archimedean $\Rightarrow$ semisimple (if finitely generated)

- A/I is simple iff I is maximal.
- A/I is semisimple iff I is intersection of maximal ℓ-ideals.

MV-algebras

Definition (Chang 1958)

An *MV-algebra* is a structure $(A, \oplus, \neg, 0)$ satisfying

1. $x \oplus (y \oplus z) = (x \oplus y) \oplus z$
2. $x \oplus y = y \oplus z$
3. $x \oplus 0 = x$
4. $\neg\neg x = x$
5. $x \oplus \neg 0 = \neg 0$
6. $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$

Riesz MV-algebras

Definition

A Riesz MV-algebra is a structure $(R, \cdot)$ where R is an MV-algebra and $\cdot : [0,1] \times R \to R$ is such that

1. If $x \odot y = 0$, then $(rx) \odot (ry) = 0$ and $r(x \oplus y) = rx \oplus ry$.
2. If $r \odot q = 0$, then $(rx) \odot (qx) = 0$ and $(r \oplus q)x = rx \oplus ry$.
3. $(rq)x = r(qx)$.
4. $1x = x$.