Characterization of metrizable Esakia spaces via some forbidden configurations

Luca Carai

joint work with Guram Bezhanishvili

New Mexico State University

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Priestley spaces

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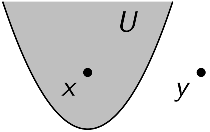
- (X, τ) is a compact topological space;
- $(X, \leq)$ is a poset;
- (Priestley separation axiom)
  for all $x, y \in X$ if $x \not\leq y$ then there is a clopen upset U such that $x \in U$ and $y \notin U$.



Priestley duality

There is a dual equivalence between the category of bounded distributive lattices and the category of Priestley spaces and continuous order-preserving maps.

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To any bounded distributive lattice L it is associated the set X of its prime filters ordered by inclusion and endowed with the topology generated by the basis $\{\beta(a) \setminus \beta(b) \mid a, b \in L\}$. Where $\beta(a) = \{x \in X \mid a \in x\}$.
To any Priestley space X it is associated the lattice L of its clopen upsets.

Esakia spaces and Esakia duality

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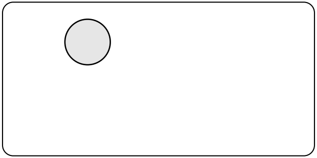

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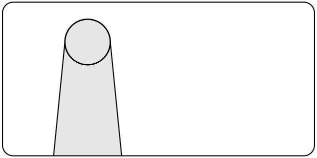

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Definition

A Priestley space is called an Esakia space if the downset generated by every clopen is clopen. Or equivalently, if the downset generated by every open is open.

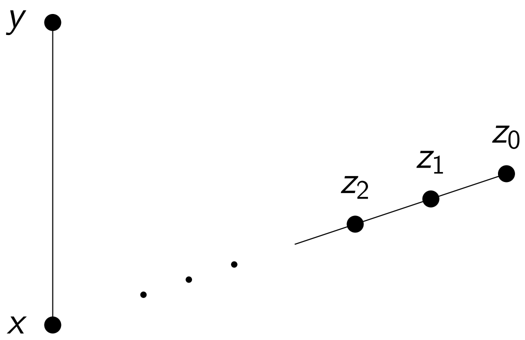
Esakia duality is a restricted version of Priestley duality: it states the existence of a dual equivalence between the category of Heyting algebras and the category of Esakia spaces and continuous p-morphisms.

$$\mathbf{Hey} \overset{d}{\cong} \mathbf{Esa}$$

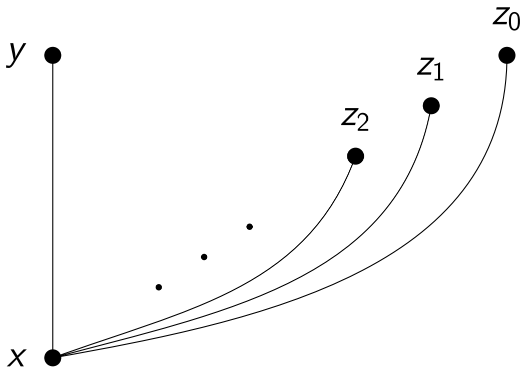
Priestley that are not Esakia

We want to give a characterization of Priestley spaces that are not Esakia by looking at their subspaces. We consider three simple Priestley spaces that are not Esakia that we call Z_1, Z_2, Z_3.

The space Z_1



The space Z_2



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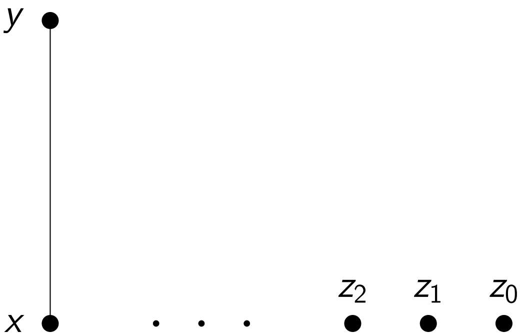
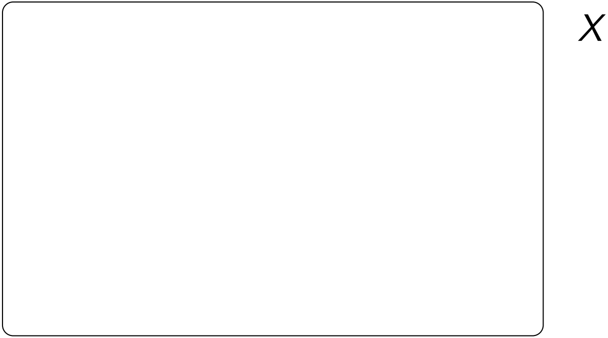


Forbidden configurations

Definition

Let X be a Priestley space. We say that Z_i ($i = 1, 2, 3$) is a forbidden configuration for X if



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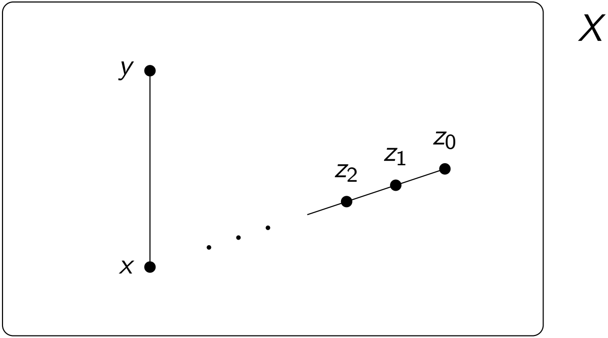
- there is a topological and order embedding $e : Z_i \to X$ and



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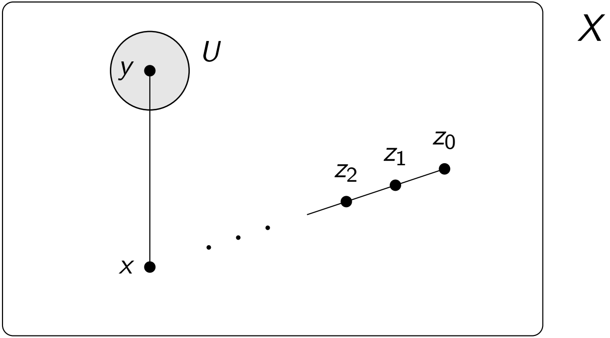

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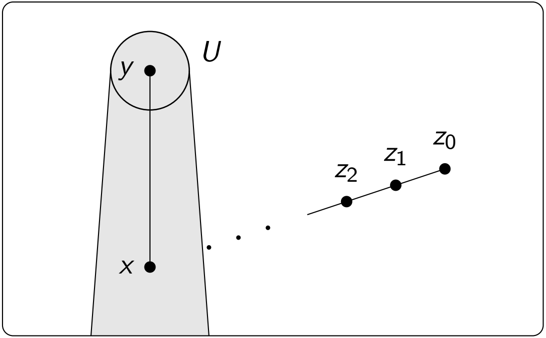
- there is a topological and order embedding $e : Z_i \to X$ and
- there is an open neighborhood U of $e(y)$ such that $e^{-1}(\downarrow U) = \{x, y\}$.



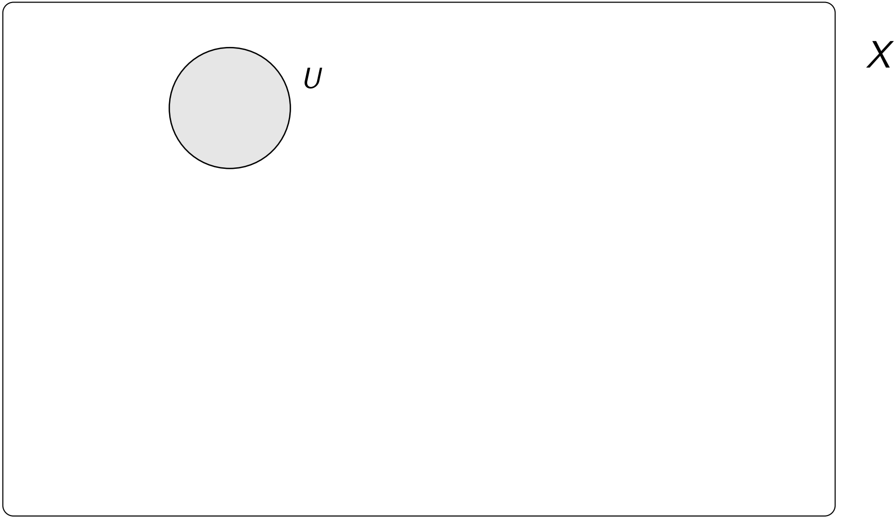
X

Main result

Theorem

A metrizable Priestley space X is not an Esakia space iff one of Z_1, Z_2, Z_3 is a forbidden configuration for X.

Proof of the main result



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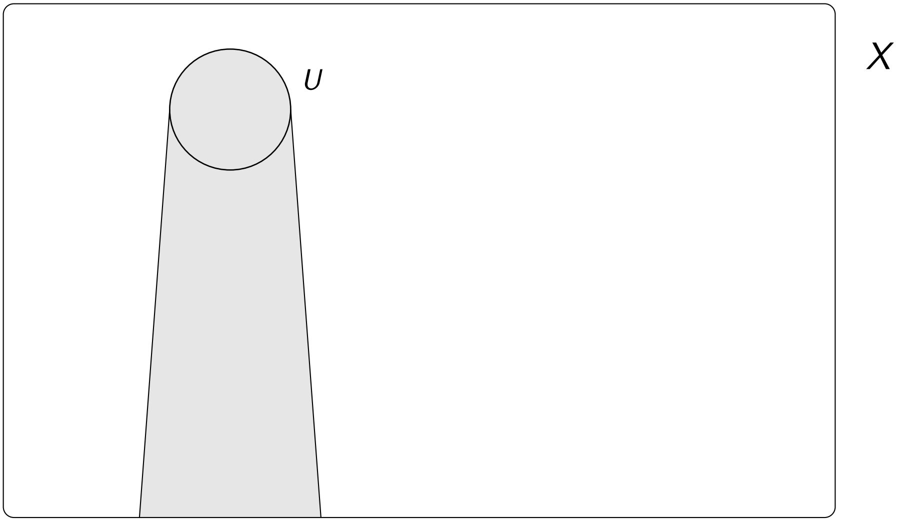

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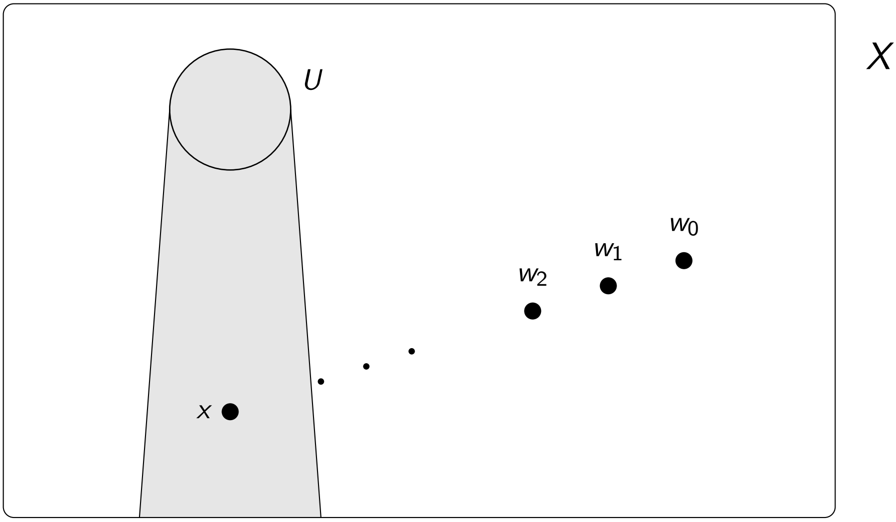

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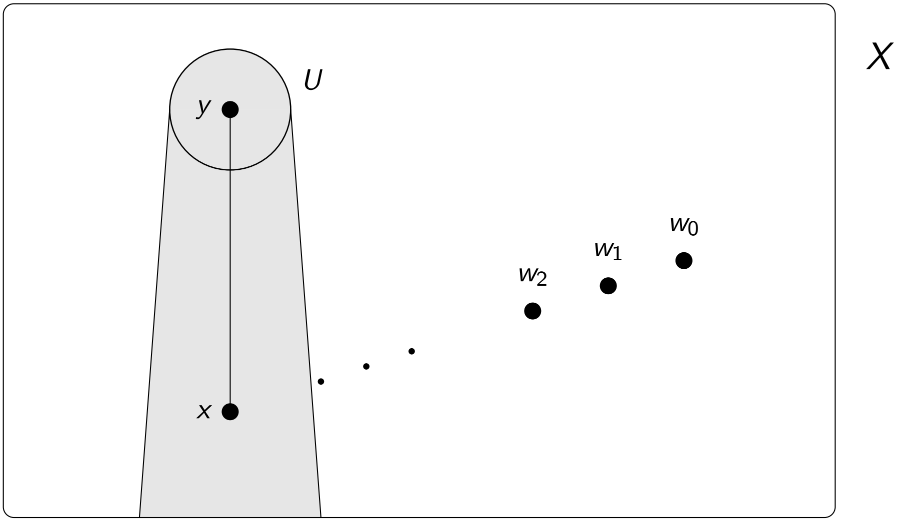
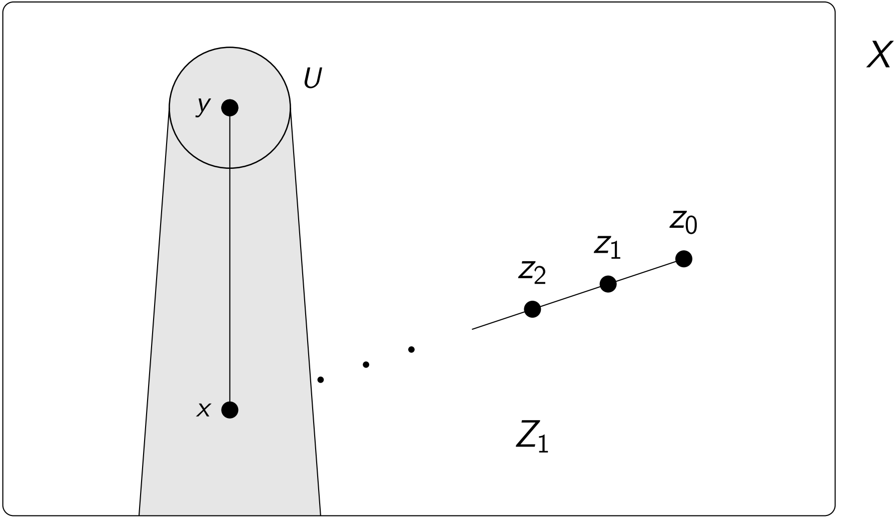


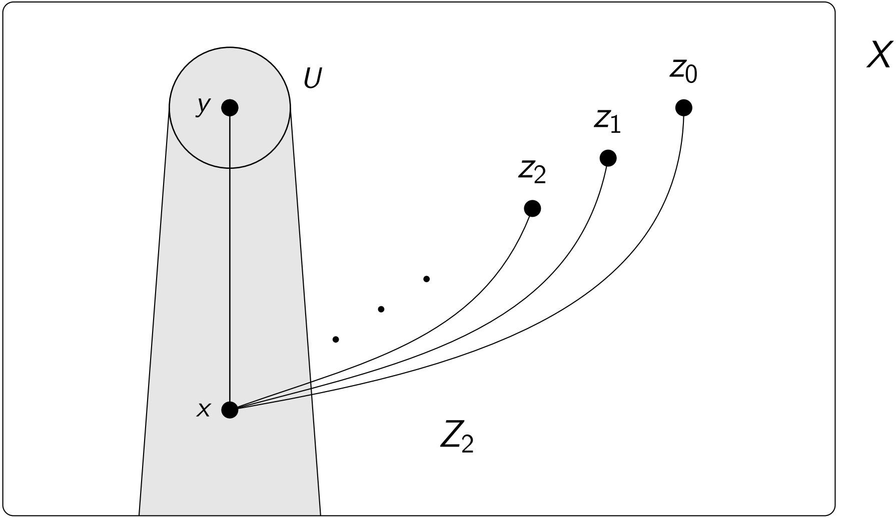
Proof of the main result

Case 1: $\{w_i\}$ contains an infinite chain.



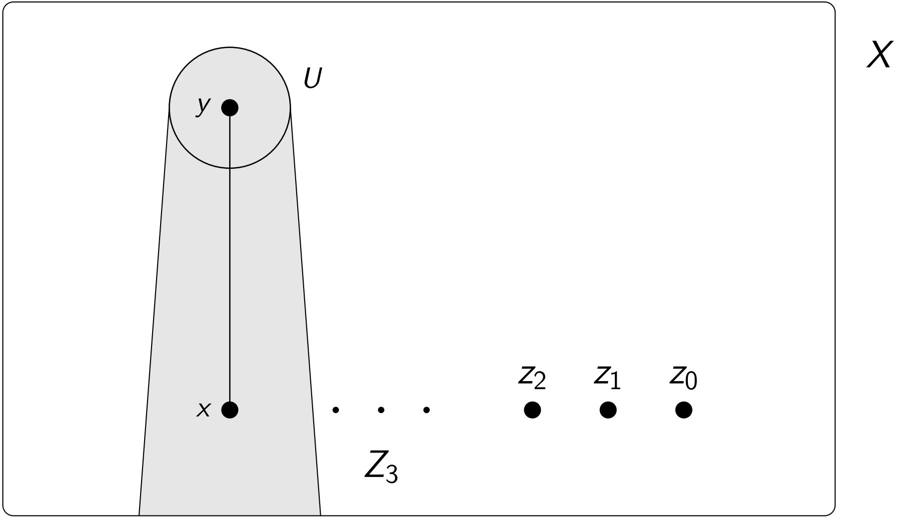
Proof of the main result

Case 2a: $\{w_i\}$ contains an infinite anti-chain above x.



Proof of the main result

Case 2b: $\{w_i\}$ contains an infinite anti-chain not above x.



Lattice-theoretic statement

We now want to translate this theorem to its lattice-theoretic dual statement.

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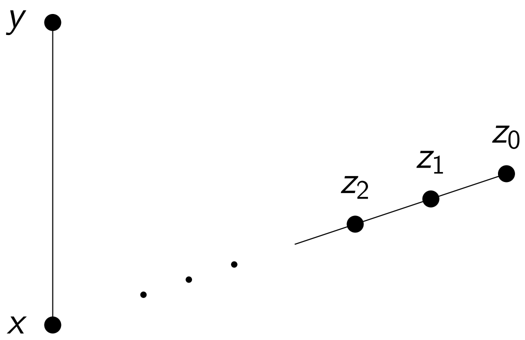
The space Z_1



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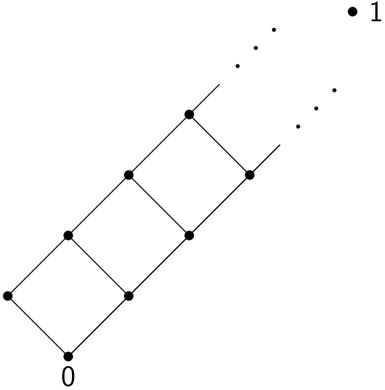

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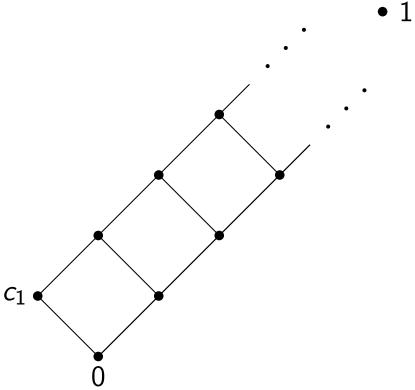
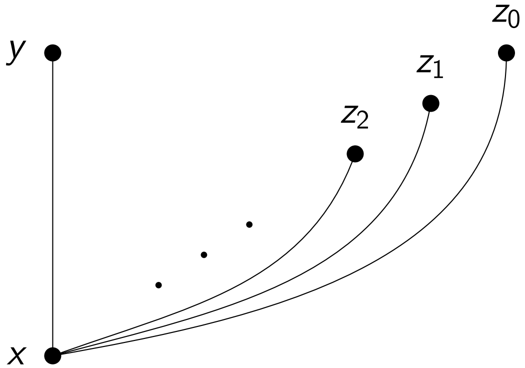


$\neg c_1 = c_1 \to 0$ doesn't exist.

The space Z_2



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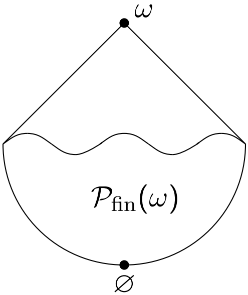

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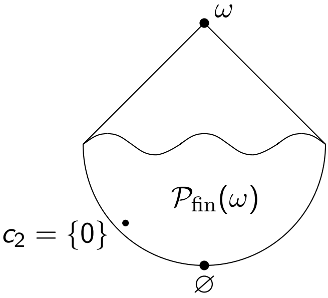

$\neg c_2 = c_2 \to \varnothing$ doesn't exist.

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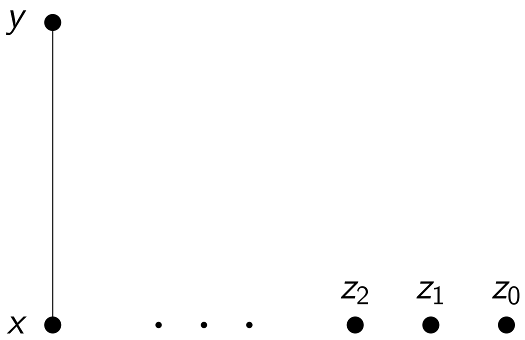

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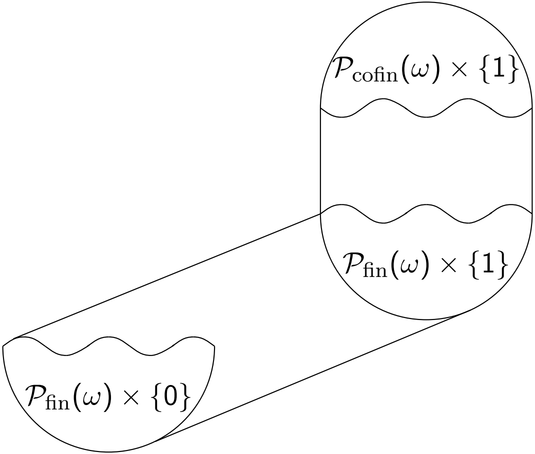

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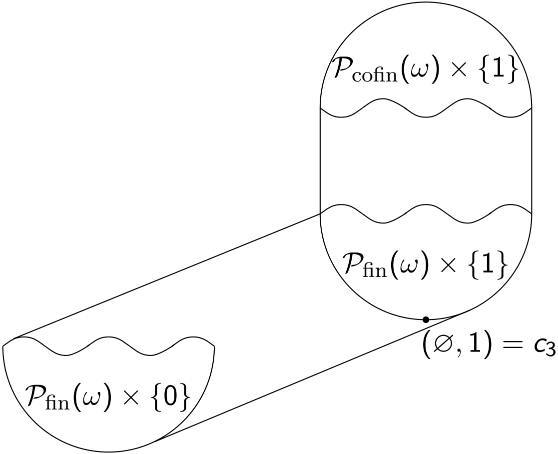

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Metrizability and countability

Proposition

Let L be a bounded distributive lattice and X its dual Priestley space. X is metrizable if and only if L is countable.

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Thus, if X is a metrizable Priestley space, then it is second countable. This implies that every clopen is a finite union of elements from the countable basis. In particular the set of clopen upsets, and so L, is countable.
On the other hand, if L is countable then X has countably many clopen upsets and therefore countably many clopen subsets which are a basis. So X is second countable and hence metrizable.

□

Algebraic dual statement of the main result

Definition

Let L be a bounded distributive lattice and $a, b \in L$. Let $I_{a \to b}$ be the ideal

$$I_{a \to b} := \{c \in L \mid c \wedge a \leq b\}$$

Remark

$I_{a \to b}$ is principal iff $a \to b$ exists in L and, in that case, $I_{a \to b} = {\downarrow}(a \to b)$.

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Theorem

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There are $a, b \in L$ such that $h_i(I_{a \to b}) = I_{c_i \to 0}$, where c_1, c_2, c_3 are the elements described above.

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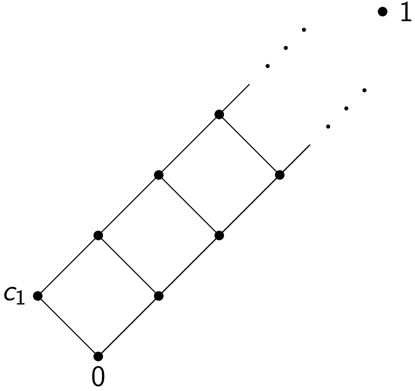

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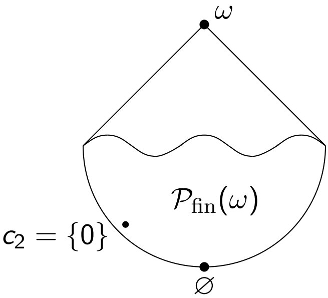

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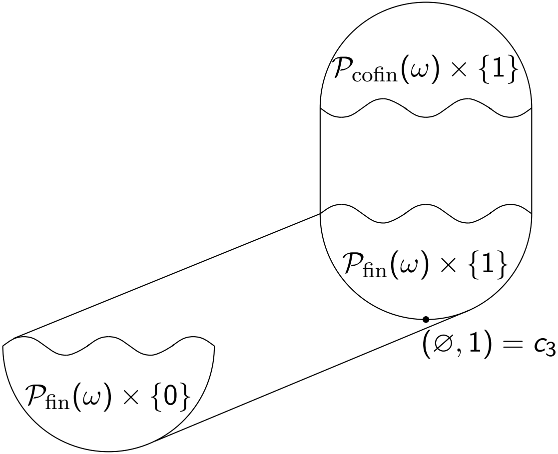

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Let X be a Priestley space. We say that Z_i $(i = 1, 2, 3)$ is a p-configuration for X if Z_i is a forbidden configuration for X and in addition the open neighborhood U of $e(y)$ is an upset.

Corollary

Let X be a metrizable Priestley space. X is not a p-space iff one of Z_1, Z_2, Z_3 is a p-configuration for X.

Characterization of duals of co-Heyting algebras

We recall that co-Heyting algebras are order-duals of Heyting algebras. The Priestley spaces dual to co-Heyting algebras are the ones with the property that the upset of each clopen is clopen.

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Corollary

A metrizable Priestley space X is not the dual of a co-Heyting algebra iff there are a topological and order embedding e from one of Z_1^*, Z_2^*, Z_3^* into X and an open neighborhood U of $e(y)$ such that $e^{-1}(\uparrow U) = \{x, y\}$.

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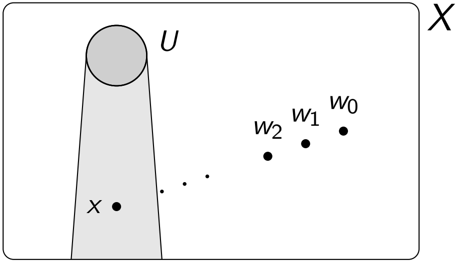
Corollary

A metrizable Priestley space X is not dual to a bi-Heyting algebra iff one of Z_1, Z_2, Z_3 is a forbidden configuration for X or there are a topological and order embedding e from one of Z_1^*, Z_2^*, Z_3^* into X and an open neighborhood U of $e(y)$ such that $e^{-1}(\uparrow U) = \{x, y\}$.

Counterexamples

In the proof we used the metrizability hypothesis only to find a sequence $\{w_i\}$ contained in the complement of $\downarrow U$ converging to $x \in \downarrow U \setminus U$.



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Note that for this step we actually just need X to be a sequential space.

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A space X is called sequential if every non-closed subset contains a sequence converging outside it.

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The following examples show that the metrizability (sequentiality) hypothesis cannot be dropped.

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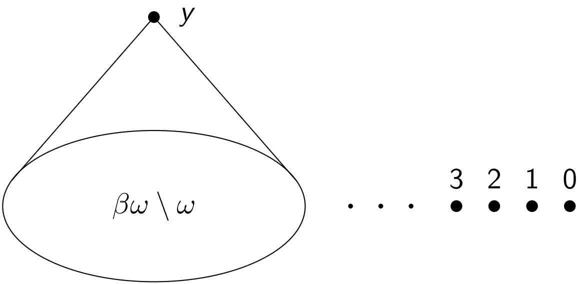

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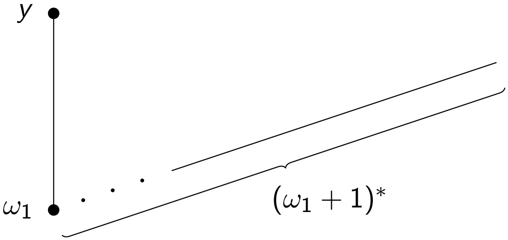

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Dually this means that every Priestley space embeds into an Esakia space. More precisely, if the Priestley space is dual to the lattice L, then X embeds into 2^L where 2 is the poset $\{0 < 1\}$ with the discrete topology. The space 2^L is endowed with the product topology and the product order and it is an Esakia space. The embedding maps $x \in X$ to the element of 2^L corresponding to the set of clopen upsets containing x.

Counterexamples

Therefore there is no way to characterize Esakia spaces by forbidding embeddings of some set of Priestley spaces. We really need that additional condition on the embedding.

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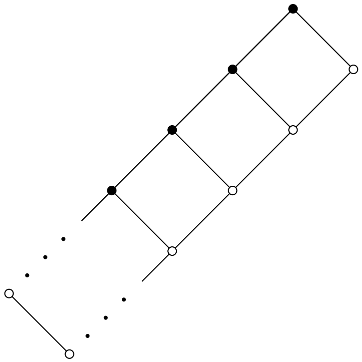
Therefore there is no way to characterize Esakia spaces by forbidding embeddings of some set of Priestley spaces. We really need that additional condition on the embedding.

The space 2^L we just described is really big and complex for almost every Priestley space X. In the following slides we present much simpler examples of Esakia spaces into which Z_1 and Z_3 embed but for which they are not forbidden configurations.

Counterexamples



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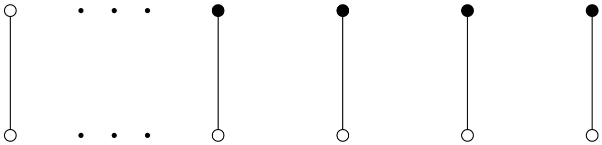

Open problems

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An obvious direction of the investigation would be trying to generalize the theorem to the non-sequential case. It seems that the problem gets really complex.

Thanks for your attention!