Modal operators on rings of continuous functions

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joint work with

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The algebraic structures having these properties are called bounded archimedean ℓ-algebras, $\boldsymbol{ba\ell}$-algebras for short.

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How can we characterize the ones isomorphic to $C(X)$ for some X?

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Any uniformly complete $\boldsymbol{ba\ell}$-algebra A is isomorphic to $C(Y_A)$.
Y_A is the Yosida space of A, the set of maximal ℓ-ideals of A with the Zariski topology. Y_A is a compact Hausdorff space.

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Theorem (Gelfand-Naimark-Stone duality)

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They provide a natural generalization of continuous functions.

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We also define $\Diamond_R f = 1 - \Box_R(1 - f)$.

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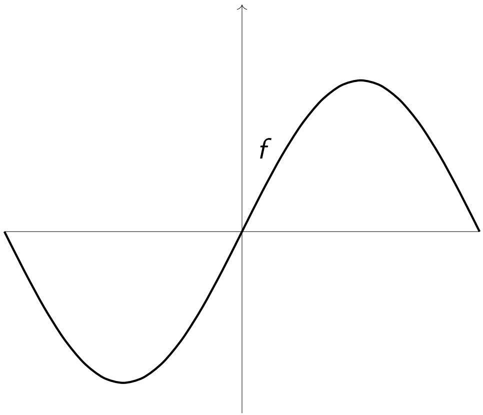

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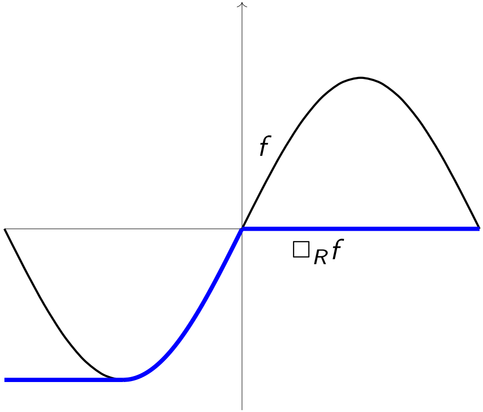

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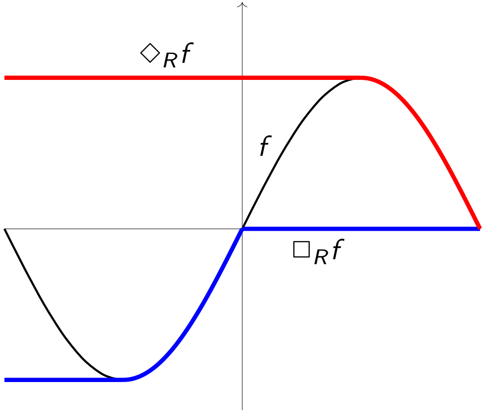
If R is a partial order, then
$\Box_R f$ is the greatest increasing function below f,
$\Diamond_R f$ the least decreasing function above f.



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(M2) $\Box\lambda = \lambda + (1 - \lambda)\Box 0$

(M3) $\Box(a^+) = (\Box a)^+$

(M4) $\Box(a + \lambda) = \Box a + \Box\lambda - \Box 0$

(M5) $\Box(\lambda a) = (\Box\lambda)(\Box a)$ provided $\lambda \geq 0$

where $a^+ = a \vee 0$.

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We proved that $R_\Box$ is a continuous relation on Y_A. This is technically rather challenging, and requires a careful study of positive parts of ℓ-ideals.

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Theorem

$\boldsymbol{muba\ell}$ *is dually equivalent to* KHF.

Esakia-Goldblatt duality

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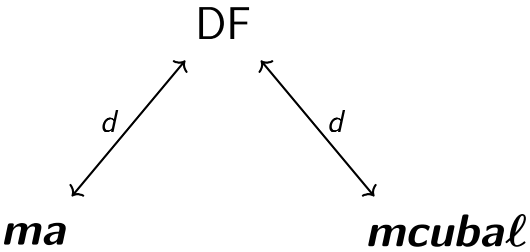

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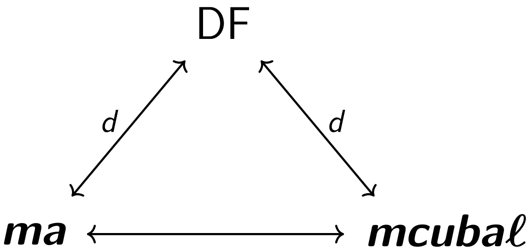

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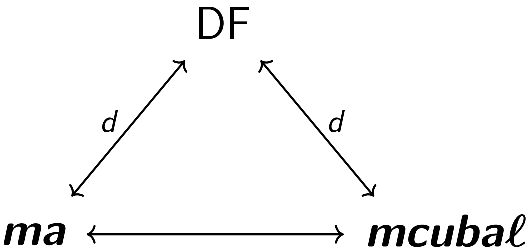

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This provides a new prospective on Esakia-Goldblatt duality.

Correspondence theory

seriality	$\Box 0 = 0$
reflexivity	$\Box a \leq a$
transitivity	$\Box a \leq \Box(\Box a(1 - \Box 0) + a\Box 0)$
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Thanks for your attention!