

Admissibility of Π_2 -Inference Rules: interpolation, model completion, and contact algebras

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Π_2 -rules

Definition

An inference rule ρ is a Π_2 -rule if it is of the form

$$\frac{F(\underline{\phi}/\underline{x}, \underline{y}) \rightarrow \chi}{G(\underline{\phi}/\underline{x}) \rightarrow \chi}$$

where $F(\underline{x}, \underline{y})$, $G(\underline{x})$ are propositional formulas.

We say that θ is obtained from ψ by an application of the rule ρ if

$$\psi = F(\underline{\phi}/\underline{x}, \underline{y}) \rightarrow \chi \quad \text{and} \quad \theta = G(\underline{\phi}/\underline{x}) \rightarrow \chi,$$

where $\underline{\phi}$ is a tuple of formulas, χ is a formula, and \underline{y} is a tuple of propositional letters not occurring in $\underline{\phi}$ and χ .

Let \mathcal{S} be a propositional modal system. We say that the rule ρ is **admissible** in \mathcal{S} if $\vdash_{\mathcal{S}+\rho} \phi$ implies $\vdash_{\mathcal{S}} \phi$ for each formula ϕ .

First method

Conservative extensions

We say that $\phi(\underline{x}) \wedge \psi(\underline{x}, \underline{y})$ is a **conservative extension** of $\phi(\underline{x})$ in \mathcal{S} if

$$\vdash_{\mathcal{S}} \phi(\underline{x}) \wedge \psi(\underline{x}, \underline{y}) \rightarrow \chi(\underline{x}) \text{ implies } \vdash_{\mathcal{S}} \phi(\underline{x}) \rightarrow \chi(\underline{x})$$

for every formula $\chi(\underline{x})$.

Theorem

If \mathcal{S} has the interpolation property, then a Π_2 -rule ρ is admissible in \mathcal{S} iff $G(\underline{x}) \wedge F(\underline{x}, \underline{y})$ is a conservative extension of $G(\underline{x})$ in \mathcal{S} .

Therefore, if \mathcal{S} has the interpolation property and conservativity is decidable in \mathcal{S} , then Π_2 -rules are effectively recognizable in \mathcal{S} .

Corollary

The admissibility problem for Π_2 -rules is

- *CONEXPTIME-complete in K and S5;*
- *in EXPSpace and NEXPTIME-hard in S4.*

Second method

Uniform interpolants

An S5-modality $[\forall]$ is called a **universal modality** if

$$\vdash_{\mathcal{S}} \bigwedge_{i=1}^n [\forall](\phi_i \leftrightarrow \psi_i) \rightarrow (\Box[\phi_1, \dots, \phi_n] \leftrightarrow \Box[\psi_1, \dots, \psi_n])$$

for every modality \Box of \mathcal{S} .

If $\phi(\underline{x}, \underline{y})$ is a formula, its **right global uniform pre-interpolant** $\forall_{\underline{x}}\phi(\underline{y})$ is a formula such that for every $\psi(\underline{y}, \underline{z})$ we have that

$$\psi(\underline{y}, \underline{z}) \vdash_{\mathcal{S}} \phi(\underline{x}, \underline{y}) \quad \text{iff} \quad \psi(\underline{y}, \underline{z}) \vdash_{\mathcal{S}} \forall_{\underline{x}}\phi(\underline{y}).$$

Theorem

Suppose that \mathcal{S} has uniform global pre-interpolants and a universal modality $[\forall]$. Then a Π_2 -rule ρ is admissible in \mathcal{S} iff

$$\vdash_{\mathcal{S}} [\forall]\forall_{\underline{y}}(F(\underline{x}, \underline{y}) \rightarrow z) \rightarrow (G(\underline{x}) \rightarrow z).$$

Third method

Simple algebras and model completions

To a Π_2 -rule we associate the first-order formula

$$\Pi(\rho) := \forall \underline{x}, z \left(G(\underline{x}) \not\leq z \Rightarrow \exists \underline{y} : F(\underline{x}, \underline{y}) \not\leq z \right).$$

Theorem

Suppose that \mathcal{S} has a universal modality. A Π_2 -rule ρ is admissible in \mathcal{S} iff for each simple \mathcal{S} -algebra \mathcal{B} there is a simple \mathcal{S} -algebra \mathcal{C} such that \mathcal{B} is a subalgebra of \mathcal{C} and $\mathcal{C} \models \Pi(\rho)$.

In the presence of a universal modality, an \mathcal{S} -algebra is simple iff

$$[\forall]x = \begin{cases} 1 & \text{if } x = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, \mathcal{S} -algebras form a **discriminator variety**. Therefore, the variety of \mathcal{S} -algebras is generated by the simple \mathcal{S} -algebras.

The **model completion** of a universal first-order theory T , if it exists, is the theory of the **existentially closed** models of T .

Let T be a universal theory in a finite language. If T is **locally finite and has the amalgamation property**, then it admits a model completion.

Theorem

Suppose that \mathcal{S} has a universal modality and let $T_{\mathcal{S}}$ be the first-order theory of the simple \mathcal{S} -algebras. If $T_{\mathcal{S}}$ has a model completion $T_{\mathcal{S}}^$, then a Π_2 -rule ρ is admissible in \mathcal{S} iff $T_{\mathcal{S}}^* \models \Pi(\rho)$ where*

$$\Pi(\rho) := \forall \underline{x}, z \left(G(\underline{x}) \not\leq z \Rightarrow \exists \underline{y} : F(\underline{x}, \underline{y}) \not\leq z \right).$$

The symmetric strict implication calculus and contact algebras

Definition (G. Bezhanishvili, N. Bezhanishvili, T. Santoli, Y. Venema (2019))

The **symmetric strict implication calculus** S^2IC is given by the axioms

$$(A0) \quad [\forall]\phi \leftrightarrow (\top \rightsquigarrow \phi),$$

$$(A1) \quad (\perp \rightsquigarrow \phi) \wedge (\phi \rightsquigarrow \top),$$

$$(A2) \quad [(\phi \vee \psi) \rightsquigarrow \chi] \leftrightarrow [(\phi \rightsquigarrow \chi) \wedge (\psi \rightsquigarrow \chi)],$$

$$(A3) \quad [\phi \rightsquigarrow (\psi \wedge \chi)] \leftrightarrow [(\phi \rightsquigarrow \psi) \wedge (\phi \rightsquigarrow \chi)],$$

$$(A4) \quad (\phi \rightsquigarrow \psi) \rightarrow (\phi \rightarrow \psi),$$

$$(A5) \quad (\phi \rightsquigarrow \psi) \leftrightarrow (\neg\psi \rightsquigarrow \neg\phi),$$

$$(A8) \quad [\forall]\phi \rightarrow [\forall][\forall]\phi,$$

$$(A9) \quad \neg[\forall]\phi \rightarrow [\forall]\neg[\forall]\phi,$$

$$(A10) \quad (\phi \rightsquigarrow \psi) \leftrightarrow [\forall](\phi \rightsquigarrow \psi),$$

$$(A11) \quad [\forall]\phi \rightarrow (\neg[\forall]\phi \rightsquigarrow \perp),$$

and modus ponens (for \rightarrow) and necessitation (for $[\forall]$).

An open subset A of a topological space is called **regular open** if $A = \text{int}(\text{cl}(A))$.

Let v be a valuation into a topological space X that maps each propositional variable to a regular open of X . We can extend each valuation on all formulas as follows

$$v(\perp) = \emptyset$$

$$v(\top) = X$$

$$v(\phi \wedge \psi) = v(\phi) \cap v(\psi)$$

$$v(\phi \vee \psi) = \text{int}(\text{cl}(v(\phi) \cup v(\psi)))$$

$$v(\neg\phi) = \text{int}(X \setminus v(\phi))$$

$$v(\phi \rightsquigarrow \psi) = \begin{cases} X & \text{if } \text{cl}(v(\phi)) \subseteq v(\psi), \\ \emptyset & \text{otherwise.} \end{cases}$$

Theorem (G. Bezhanishvili, N. Bezhanishvili, T. Santoli, Y. Venema (2019))

$\vdash_{S2IC} \phi$ iff $v(\phi) = X$ for every compact Hausdorff space X and v .

The algebras associated with S^2IC are called **strict implication algebras**.

When a strict implication algebra is simple, \rightsquigarrow becomes a characteristic function of a binary relation. They correspond exactly to contact algebras.

Definition

A **contact algebra** is a boolean algebra equipped with a binary relation \prec satisfying the axioms:

- (S1) $0 \prec 0$ and $1 \prec 1$;
- (S2) $a \prec b, c$ implies $a \prec b \wedge c$;
- (S3) $a, b \prec c$ implies $a \vee b \prec c$;
- (S4) $a \leq b \prec c \leq d$ implies $a \prec d$;
- (S5) $a \prec b$ implies $a \leq b$;
- (S6) $a \prec b$ implies $\neg b \prec \neg a$.

Theorem

The model completion Con^ of the theory of contact algebras is finitely axiomatizable.*

An axiomatization is given by the following three sentences.

$$\forall a, b_1, b_2 (a \neq 0 \ \& \ (b_1 \vee b_2) \wedge a = 0 \ \& \ a \prec a \vee b_1 \vee b_2 \Rightarrow \\ \exists a_1, a_2 (a_1 \vee a_2 = a \ \& \ a_1 \wedge a_2 = 0 \ \& \ a_1 \neq 0 \ \& \ a_2 \neq 0 \ \& \ a_1 \prec a_1 \vee b_1 \\ \& \ a_2 \prec a_2 \vee b_2))$$

$$\forall a, b (a \wedge b = 0 \ \& \ a \not\prec \neg b \Rightarrow \exists a_1, a_2 (a_1 \vee a_2 = a \ \& \ a_1 \wedge a_2 = 0 \\ \& \ a_1 \not\prec \neg b \ \& \ a_2 \not\prec \neg b \ \& \ a_1 \prec \neg a_2))$$

$$\forall a (a \neq 0 \Rightarrow \exists a_1, a_2 (a_1 \vee a_2 = a \ \& \ a_1 \wedge a_2 = 0 \ \& \ a_1 \prec a \ \& \ a_1 \not\prec a_1))$$

The following Π_2 -rule

$$\frac{(p \rightsquigarrow p) \wedge (\phi \rightsquigarrow p) \wedge (p \rightsquigarrow \psi) \rightarrow \chi}{(\phi \rightsquigarrow \psi) \rightarrow \chi}$$

corresponds to the zero-dimensionality of the space.

Using the axiomatization of Con^* it is easy to show that it is admissible in $S^2\text{IC}$.

Therefore, $S^2\text{IC}$ is complete wrt Stone spaces.

Theorem (G. Bezhanishvili, N. Bezhanishvili, T. Santoli, Y. Venema (2019))

$\vdash_{S^2\text{IC}} \phi$ iff $v(\phi) = X$ for every Stone space X and v .

THANK YOU!

We stand with Ukraine!