Deriving Priestley and Esakia dualities and their generalizations from Pontryagin duality for semilattices

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Pontryagin duality for semilattices

Pontryagin and Stone dualities are two important results that date back to the 1930s.

- Pontryagin duality states that the category of locally compact Abelian groups is self-dual.
- Stone duality establishes a dual equivalence between the categories of boolean algebras and of zero-dimensional compact Hausdorff spaces (Stone spaces).
- A corollary of Pontryagin duality yields that the categories of torsion Abelian groups and of Stone Abelian groups are dually equivalent.
- Hofmann, Mislove, and Stralka in 1974 developed a version of Pontryagin duality for semilattices:

Theorem (Pontryagin duality for semilattices)

The category **MS** *of (meet-)semilattices is dually equivalent to the category of Stone (meet-)semilattices.*

Algebraic lattices

Theorem

The category of Stone semilattices is dually isomorphic to the category of algebraic lattices.

Definition

- An element a of a complete lattice is compact if $a \leq \bigvee S$ implies $a \leq \bigvee T$ for some finite $T \subseteq S$.
- An algebraic lattice is a complete lattice in which every element is a join of compact elements.

Algebraic lattices play an important role in universal algebra.

Theorem

Algebraic lattices are (up to isomorphism) the lattices of congruences of algebras.

Semilattices and algebraic lattices

Let **AlgLat** be the category of algebraic lattices and maps preserving arbitrary joins and compact elements.

Corollary

MS *is equivalent to* **AlgLat**.

Filt: **MS** $\to$ **AlgLat**

- $M \in$ **MS** is sent to $\mathrm{Filt}(M) = \{\text{filters of } M\}$ ordered by $\subseteq$;
- If $\alpha \colon M_1 \to M_2$, then $\mathrm{Filt}(\alpha)(F)$ is the filter generated by $\alpha[F]$.

K: **AlgLat** $\to$ **MS**

- $L \in$ **AlgLat** is sent to $\mathrm{K}(L) = \{\text{compact elements of } L\}$ ordered by $\geq$;
- If $f \colon L_1 \to L_2$, then $\mathrm{K}(f) = f_{|\mathrm{K}(L_1)}$.

A version of this correspondence between semilattices and algebraic lattices already appeared in the works of Nachbin from the 1940s.

Goals:

- restrict the equivalence between **MS** and **AlgLat** to subcategories of **MS**,
- connect such equivalences with the well-known Priestley and Esakia dualities and their generalizations.

Since these equivalences involve categories of frames, this investigation yields a frame-theoretic perspective of Priestley and Esakia dualities.

Pontryagin for distributive lattices

Definition

- A complete lattice is called a frame if finite meets distribute over arbitrary joins $a \wedge \bigvee S = \bigvee \{a \wedge s \mid s \in S\}$.
- A coherent frame is a frame L that is an algebraic lattice and $\mathrm{K}(L)$ forms a bounded sublattice of L.
- A coherent frame homomorphism is a frame homomorphisms that preserves compact elements.

Theorem

The functors Filt *and* K *restrict to a equivalence between the categories of (bounded) distributive lattices and of coherent frames.*

Priestley spaces

Definition

- A Priestley space is a Stone space equipped with a partial order $\leq$ that satisfies the Priestley separation axiom:
  If $x \nleq y$, then there is U clopen upset such that $x \in U$ and $y \notin U$.
- A morphism of Priestley spaces is a continuous map that is order preserving.

Priestley duality

Theorem (Priestley 1972)

The category **PS** *of Priestley spaces and the category* **DL** *of distributive lattices are dually equivalent.*

$\mathrm{ClopUp}\colon \mathbf{PS} \to \mathbf{DL}$

- $X \in \mathbf{PS}$ is sent to $\mathrm{ClopUp}(X) = \{\text{clopen upsets of } X\}$;
- If $f\colon X_1 \to X_2$, then $\mathrm{ClopUp}(f)(U) = f^{-1}(U)$.

$\mathrm{Spec}\colon \mathbf{DL} \to \mathbf{PS}$

- $L \in \mathbf{DL}$ is sent to $\mathrm{Spec}(L) = \{\text{prime filters of } L\}$ ordered by $\subseteq$ and topologized by the patch topology of the spectral topology;
- If $\alpha\colon L_1 \to L_2$, then $\mathrm{Spec}(\alpha)(P) = \alpha^{-1}(P)$.

PS and CohFrm

We established a direct way to obtain the dual equivalence between **PS** and **CohFrm**.

Theorem (Bezhanishvili, C., Morandi 2022)

A dual equivalence between **PS** *and* **CohFrm** *is established by the following contravariant functors:*

$\mathrm{ClUp}\colon \mathbf{PS} \to \mathbf{CohFrm}$

- *X is sent to the frame $\mathrm{ClUp}(X) = \{$closed upsets of $X\}$ ordered by reverse inclusion;*
- *If $f: X_1 \to X_2$, then $\mathrm{ClUp}(f)(C) = f^{-1}(C)$.*

$\mathrm{P}\colon \mathbf{CohFrm} \to \mathbf{PS}$

- *$L \in$ **CohFrm** is sent to $\mathrm{P}(L) = \{$meet-prime elements of $L\}$ ordered by $\leq$. Since $P(L)$ is the set of points of the frame L, we consider the patch topology of the the natural topology on $P(L)$.*
- *If $\alpha\colon L_1 \to L_2$, then $P(\alpha) = r_{|P(L_1)}$, where r is the right adjoint of α.*

Distributive semilattices and algebraic frames

Definition

A (meet-)semilattice is called distributive if $a \wedge b \leq c$ implies there are $a' \geq a$ and $b' \geq b$ s.t. $a' \wedge b' = c$.

A lattice $(L, \wedge, \vee)$ is distributive iff $(L, \wedge)$ is distributive as a semilattice. Note that **DSLat** is **not** a variety.

Definition

- An algebraic frame is an algebraic lattice that is a frame.
- Let $\mathbf{AlgFrm_J}$ be the category algebraic frames and morphisms that preserve arbitrary joins and compact elements.

Since an algebraic lattice L is a frame iff it is distributive iff $\mathrm{K}(L)$ is a distributive semilattice, we obtain:

Theorem

The category of distributive semilattices is equivalent to $\mathbf{AlgFrm_J}$.

Pointed generalized Priestley spaces

Definition

A pointed generalized Priestley space (X, X_0) is a Priestley space X with maximum m together with a dense subspace X_0 of $X \setminus \{m\}$ such that:

- $\downarrow X_0 = X \setminus \{m\}$;
- the Priestley separation axiom holds relative to $\mathcal{A}(X)$: if $x \not\leq y$, there exists $U \in \mathcal{A}(X)$ s.t. $x \in U$ and $y \notin U$;
- $X_0 = \{x \in X : \mathcal{I}_x$ is nonempty and updirected$\}$.

where $\mathcal{A}(X)$ is the set of clopen upsets U such that $\max(X \setminus U) \subseteq X_0$, and $\mathcal{I}_x = \{U \in \mathcal{A}(X) \mid x \notin U\}$

Definition

A generalized Priestley morphism between X, Y pointed generalized Priestley spaces is a relation $R \subseteq X \times Y$ such that

- If $x \not R\, y$, then there is $U \in \mathcal{A}(Y)$ such that $R[x] \subseteq U$ and $y \notin U$;
- If $U \in \mathcal{A}(Y)$, then $X \setminus R^{-1}[Y \setminus U] \in \mathcal{A}(X)$.

Generalized Priestley duality

Let **PGPS** be the category of pointed generalized Priestley spaces and generalized Priestley morphisms.

Theorem (Bezhanishvili and Jansana 2008)

PGPS *is dually equivalent to the category* **DSLat** *of distributive semilattices.*

$\mathcal{A}$: **PGPS** $\to$ **DSLat**

- $(X, X_0) \in$ **PGPS** is sent to $\mathcal{A}(X) = \{$admissible clopen upsets of $X\}$. A clopen upset U is admissible if $\max(X \setminus U) \subseteq X_0$.

$\mathcal{X}$: **DSLat** $\to$ **PGPS**

- $M \in$ **DSLat** is sent to the space $\mathcal{X}(M) = (X, X_0)$, where $X = \{$optimal filters of $M\} \cup \{M\}$ and $X_0 = \{$prime filters of $M\}$.

$\mathbf{AlgFrm_J}$ and $\mathbf{PGPS}$

Theorem (Bezhanishvili, C., Morandi 2022)

A dual equivalence between $\mathbf{PGPS}$ *and* $\mathbf{AlgFrm_J}$ *is established by the following contravariant functors:*

$\mathcal{V}^a \colon \mathbf{PGPS} \to \mathbf{AlgFrm_J}$

- $(X, X_0) \in \mathbf{PGPS}$ *is sent to* $\mathcal{V}^a(X) = \{$*admissible closed upsets of* $X\}$ *ordered by reverse inclusion.*

$\mathrm{PP} \colon \mathbf{AlgFrm_J} \to \mathbf{PGPS}$

- $L \in \mathbf{AlgFrm_J}$ *is sent to* $(\mathrm{PP}(L) \cup \{1\}, \mathrm{P}(L))$, *where* $\mathrm{PP}(L)$ *is the set of pseudoprime elements of* L.

Definition

- If $(X, X_0) \in \mathbf{PGPS}$, then a closed upset $C \subseteq X$ is admissible if $(X \setminus C) \subseteq {\downarrow}(X_0 \setminus C)$.
- If L is an algebraic frame, then $p \in L$ is pseudoprime if $a_1 \wedge \cdots \wedge a_n \ll p$ implies $a_i \leq p$ for some i.

Heyting algebras and Brouwerian semilattices

Definition

- A (meet-)semilattice M is a Brouwerian semilattice if for each $a, b \in M$ there exists $a \to b \in M$ such that $c \leq a \to b$ iff $a \wedge c \leq b$ for any $c \in M$.
- A Heyting algebra is a bounded lattice that is a Brouwerian semilattice.

- Heyting algebras form a variety that yields the algebraic semantics of the intuitionistic propositional calculus.
- Brouwerian semilattices are always distributive semilattices and form a variety, which yields the algebraic semantics of the $(\wedge, \to)$-fragment of the intuitionistic propositional calculus.

Esakia duality and its generalization

Definition

- A Priestley space is an Esakia space if U clopen implies $\downarrow U$ clopen.
- A pointed generalized Priestley space is a pointed generalized Esakia space if $U, V \in \mathcal{A}(X)$ implies $\downarrow(U \setminus V)$ clopen.

Theorem (Esakia 1974)

Priestley duality restricts to a dual equivalence between the categories of Esakia spaces and of Heyting algebras.

Theorem (Bezhanishvili and Jansana 2008)

Generalized Priestley duality restricts to a dual equivalence between the categories of pointed generalized Esakia spaces and of Brouwerian semilattices.

Heyting and Brouwerian frames

Definition

- A Heyting frame L is an algebraic frame in which $\mathsf{K}(L)$ is a Heyting subalgebra.
- A Brouwerian frame L is an algebraic frame in which $(\mathsf{K}(L), \geq)$ is a Brouwerian semilattice.

Theorem (Bezhanishvili, C., Morandi 2022)

- *The category of Heyting frames is equivalent to the one of Heyting algebras and dually equivalent to the one of Esakia spaces.*
- *The category of Brouwerian frames is equivalent to the one of Brouwerian semilattices and dually equivalent to the one of pointed generalized Esakia spaces.*

Köhler and Pigozzi in 1980 showed that a variety V has equationally definable principal congruences iff the congruences of each $A \in V$ form a Brouwerian frame. These varieties correspond to the algebraizable logics with the deduction theorem.

THANK YOU!