Temporal interpretation of intuitionistic quantifiers

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Predicate Gödel translation

$$\mathsf{IQC} \longrightarrow \mathsf{QS4}$$

$$\begin{array}{rcl}
\bot^t & = & \bot \\
P(x_1,\ldots,x_n)^t & = & \Box P(x_1,\ldots,x_n) \\
(A \wedge B)^t & = & A^t \wedge B^t \\
(A \vee B)^t & = & A^t \vee B^t \\
(A \to B)^t & = & \Box(A^t \to B^t) \\
(\forall x A)^t & = & \Box \forall x A^t \\
(\exists x A)^t & = & \exists x A^t
\end{array}$$

Tense language

We consider a tense languages containing two modalities

- $\Box_F$ interpreted as "always in the future",
- $\Box_P$ interpreted as "always in the past".

Consequently

- $\Diamond_F := \neg\Box_F\neg$ interpreted as "sometime in the future",
- $\Diamond_P := \neg\Box_P\neg$ interpreted as "sometime in the past".

S4.t

Definition

The logic S4.t is the least set of formulas of the tense propositional language containing all substitution instances of S4-axioms for both $\Box_F$ and $\Box_P$, the axiom schemes

1. $A \to \Box_P \Diamond_F A$
2. $A \to \Box_F \Diamond_P A$

and closed under the inference rules

$$\frac{A \quad A \to B}{B} \quad \text{Modus Ponens (MP)}$$

$$\frac{A}{\Box_F A} \quad \Box_F\text{-Necessitation } (\mathrm{N_F}) \qquad \frac{A}{\Box_P A} \quad \Box_P\text{-Necessitation } (\mathrm{N_P})$$

Temporal translation

$$\mathrm{IPC} \longrightarrow \mathrm{S4}$$

$$\begin{array}{rcl} \bot^t & = & \bot \\ p^t & = & \Box p \\ (A \wedge B)^t & = & A^t \wedge B^t \\ (A \vee B)^t & = & A^t \vee B^t \\ (A \rightarrow B)^t & = & \Box(A^t \rightarrow B^t) \end{array}$$

Temporal translation

$$\mathrm{IPC} \longrightarrow \mathrm{S4.t}$$

$$\begin{array}{rcl}
\bot^t & = & \bot \\
p^t & = & \Box_F p \\
(A \wedge B)^t & = & A^t \wedge B^t \\
(A \vee B)^t & = & A^t \vee B^t \\
(A \to B)^t & = & \Box_F(A^t \to B^t)
\end{array}$$

Temporal translation

$$\mathrm{HB} \longrightarrow \mathrm{S4.t}$$

$$\begin{array}{rcl}
\bot^t & = & \bot \\
p^t & = & \Box_F p \\
(A \wedge B)^t & = & A^t \wedge B^t \\
(A \vee B)^t & = & A^t \vee B^t \\
(A \to B)^t & = & \Box_F (A^t \to B^t) \\
(A \leftarrow B)^t & = &
\end{array}$$

Temporal translation

$$\mathrm{HB} \longrightarrow \mathrm{S4.t}$$

$$\begin{array}{rcl}
\bot^t & = & \bot \\
p^t & = & \Box_F p \\
(A \wedge B)^t & = & A^t \wedge B^t \\
(A \vee B)^t & = & A^t \vee B^t \\
(A \rightarrow B)^t & = & \Box_F (A^t \rightarrow B^t) \\
(A \leftarrow B)^t & = & \Diamond_P (\neg A^t \wedge B^t)
\end{array}$$

Predicate Gödel translation

$$\mathsf{IQC} \longrightarrow \mathsf{QS4}$$

$$\begin{array}{rcl}
\bot^t & = & \bot \\
P(x_1, \ldots, x_n)^t & = & \Box P(x_1, \ldots, x_n) \\
(A \wedge B)^t & = & A^t \wedge B^t \\
(A \vee B)^t & = & A^t \vee B^t \\
(A \to B)^t & = & \Box(A^t \to B^t) \\
(\forall x A)^t & = & \Box \forall x A^t \\
(\exists x A)^t & = & \exists x A^t
\end{array}$$

IQC

Let $\mathcal{L}$ be a predicate language without function symbols.

Definition

The *intuitionistic predicate logic* IQC is the least set of formulas of $\mathcal{L}$ containing all substitution instances of theorems of IPC, the axiom schemes

1. $\forall x A \to A(y/x)$ Universal instantiation (UI)
2. $A(y/x) \to \exists x A$
3. $\forall x(A \to B) \to (A \to \forall x B)$ with x not free in A
4. $\forall x(A \to B) \to (\exists x A \to B)$ with x not free in B

and closed under the inference rules

$$\frac{A \quad A \to B}{B} \ (\mathrm{MP}) \qquad \frac{A}{\forall x A} \ (\mathrm{Gen})$$

Kripke semantics for IQC

Definition

An IQC-*frame* is a triple $\mathfrak{F} = (W, R, D)$ where

- W is a nonempty set whose elements are called the *worlds* of $\mathfrak{F}$.
- R is a partial order on W.
- D is a function that associates to each $w \in W$ a nonempty set D_w such that wRv implies $D_w \subseteq D_v$ for each $w, v \in W$. The set D_w is called the *domain* of w.

Kripke semantics for IQC

Definition

- An *interpretation* of $\mathcal{L}$ in $\mathfrak{F}$ is a function I associating to each world w and any n-ary predicate symbol P an n-ary relation $I_w(P) \subseteq (D_w)^n$ such that wRv implies $I_w(P) \subseteq I_v(P)$.
- A *model* is a pair $\mathfrak{M} = (\mathfrak{F}, I)$ where $\mathfrak{F}$ is an IQC-frame and I is an interpretation in $\mathfrak{F}$.
- Let w be a world of $\mathfrak{F}$. A *w-assignment* is a function σ associating to each individual variable x an element $\sigma(x)$ of D_w. Note that if wRv, then σ is also a v-assignment.
- Let σ and τ be two w-assignments and x an individual variable. Then τ is said to be an *x-variant* of σ if $\tau(y) = \sigma(y)$ for all $y \neq x$.

Kripke semantics for IQC

Definition

$\mathfrak{M} \vDash_w^\sigma \bot$		never
$\mathfrak{M} \vDash_w^\sigma P(x_1, \ldots, x_n)$	iff	$(\sigma(x_1), \ldots, \sigma(x_n)) \in I_w(P)$
$\mathfrak{M} \vDash_w^\sigma B \wedge C$	iff	$\mathfrak{M} \vDash_w^\sigma B$ and $\mathfrak{M} \vDash_w^\sigma C$
$\mathfrak{M} \vDash_w^\sigma B \vee C$	iff	$\mathfrak{M} \vDash_w^\sigma B$ or $\mathfrak{M} \vDash_w^\sigma C$
$\mathfrak{M} \vDash_w^\sigma B \to C$	iff	for all v with wRv, if $\mathfrak{M} \vDash_v^\sigma B$, then $\mathfrak{M} \vDash_v^\sigma C$
$\mathfrak{M} \vDash_w^\sigma \forall x B$	iff	for all v with wRv and each v-assignment τ that is an x-variant of σ, $\mathfrak{M} \vDash_v^\tau B$
$\mathfrak{M} \vDash_w^\sigma \exists x B$	iff	there exists a w-assignment τ that is an x-variant of σ such that $\mathfrak{M} \vDash_w^\tau B$

Kripke semantics for IQC

Definition

- We say that A is *true* in a world w of $\mathfrak{M}$, written $\mathfrak{M} \models_w A$, if for all w-assignments σ, we have $\mathfrak{M} \models_w^\sigma A$.
- We say that A is *true* in $\mathfrak{M}$, written $\mathfrak{M} \models A$, if for all worlds $w \in W$, we have $\mathfrak{M} \models_w A$.
- We say that A is *valid* in a frame $\mathfrak{F}$, written $\mathfrak{F} \models A$, if for all models $\mathfrak{M}$ based on $\mathfrak{F}$, we have $\mathfrak{M} \models A$.

Theorem (Kripke 1965)

The intuitionistic predicate logic IQC *is sound and complete with respect to Kripke semantics; that is, for each formula A,*

$$\mathsf{IQC} \vdash A \quad \textit{iff} \quad \mathfrak{F} \models A \textit{ for each } \mathsf{IQC}\textit{-frame } \mathfrak{F}.$$

Asymmetry

$\mathfrak{M} \vDash_w^\sigma \forall x B$	iff	for all v with wRv and each v-assignment τ that is an x-variant of σ, $\mathfrak{M} \vDash_v^\tau B$ (*B is true for every object in every world accessible from w*)

Asymmetry

$\mathfrak{M} \models^{\sigma}_{w} \forall x B$	iff	for all v with wRv and each v-assignment τ that is an x-variant of σ, $\mathfrak{M} \models^{\tau}_{v} B$ (*B is true for every object in every world accessible from w*)
$\mathfrak{M} \models^{\sigma}_{w} \exists x B$	iff	there exists a w-assignment τ that is an x-variant of σ such that $\mathfrak{M} \models^{\tau}_{w} B$ (*B is true for some object in w*)

Removing asymmetry via the temporal interpretation

$\mathfrak{M} \models^{\sigma}_{w} \forall x B$	iff	for all v with wRv and each v-assignment τ that is an x-variant of σ, $\mathfrak{M} \models^{\tau}_{v} B$ (*B is true for every object in the future)*
$\mathfrak{M} \models^{\sigma}_{w} \exists x B$	iff	there exists a w-assignment τ that is an x-variant of σ such that $\mathfrak{M} \models^{\tau}_{w} B$ (*B is true for some object in the present)*

Removing asymmetry via the temporal interpretation

$\mathfrak{M} \models_w^\sigma \forall x B$	iff	for all v with wRv and each v-assignment τ that is an x-variant of σ, $\mathfrak{M} \models_v^\tau B$ (*B is true for every object in the future)*
$\mathfrak{M} \models_w^\sigma \exists x B$	iff	there exists v with vRw and a v-assignment τ that is an x-variant of σ such that $\mathfrak{M} \models_v^\tau B$ (*B is true for some object in the past)*

Temporal translation

$$\mathrm{IQC} \longrightarrow \mathrm{QS4}$$

$$\begin{array}{rcl}
\bot^t & = & \bot \\
P(x_1, \ldots, x_n)^t & = & \Box P(x_1, \ldots, x_n) \\
(A \wedge B)^t & = & A^t \wedge B^t \\
(A \vee B)^t & = & A^t \vee B^t \\
(A \rightarrow B)^t & = & \Box(A^t \rightarrow B^t) \\
(\forall x A)^t & = & \Box \forall x A^t \\
(\exists x A)^t & = & \exists x A^t
\end{array}$$

Temporal translation

$$\mathrm{IQC} \longrightarrow ???$$

$$\begin{array}{rcl}
\bot^t & = & \bot \\
P(x_1, \ldots, x_n)^t & = & \Box_F P(x_1, \ldots, x_n) \\
(A \wedge B)^t & = & A^t \wedge B^t \\
(A \vee B)^t & = & A^t \vee B^t \\
(A \to B)^t & = & \Box_F (A^t \to B^t) \\
(\forall x A)^t & = & \Box_F \forall x A^t
\end{array}$$

Temporal translation

$$\mathrm{IQC} \longrightarrow ???$$

$$\begin{array}{rcl}
\bot^t & = & \bot \\
P(x_1, \ldots, x_n)^t & = & \Box_F P(x_1, \ldots, x_n) \\
(A \wedge B)^t & = & A^t \wedge B^t \\
(A \vee B)^t & = & A^t \vee B^t \\
(A \to B)^t & = & \Box_F (A^t \to B^t) \\
(\forall x A)^t & = & \Box_F \forall x A^t \\
(\exists x A)^t & = & \Diamond_P \exists x A^t
\end{array}$$

Predicate modal logics

Let $\mathcal{L}_\Box$ be a predicate modal language with the modality $\Box$.

Definition

QK is the smallest set of formulas of $\mathcal{L}_\Box$ containing all the substitution instances of the K-theorems, the axiom schemes

1. $\forall x A \to A(y/x)$ Universal instantiation (UI)
2. $\forall x(A \to B) \to (A \to \forall x B)$ with x not free in A

and closed under (MP) and

$$\frac{A}{\Box A} \quad \text{Necessitation} \qquad \frac{A}{\forall x A} \quad \text{Generalization (Gen)}$$

Definition

Let L be a propositional normal modal logic with a single modality $\Box$.
Its *predicate extension* QL is the predicate modal logic obtained by adding to QK all the substitution instances of theorems of L.
QS4 is the predicate extension of S4.

Kripke semantics for predicate modal logics

Definition

A *predicate Kripke frame* (or QK-frame) is a triple $\mathfrak{F} = (W, R, D)$ where

- W is a nonempty set whose elements are called the *worlds* of $\mathfrak{F}$.
- R is a binary relation on W.
- D is a function that associates to each $w \in W$ a nonempty set D_w such that wRv implies $D_w \subseteq D_v$ for each $w, v \in W$. The set D_w is called the *domain* of w.

Definition

- An *interpretation* of $\mathcal{L}_\Box$ in $\mathfrak{F}$ is a function I associating to each world w and any n-ary predicate symbol P an n-ary relation $I_w(P) \subseteq (D_w)^n$.
- A *model* is a pair $\mathfrak{M} = (\mathfrak{F}, I)$ where $\mathfrak{F}$ is a QK-frame and I is an interpretation in $\mathfrak{F}$.
- Assignments and x-variants are defined like for IQC-frames.

Kripke semantics for predicate modal logics

Connectives and quantifiers are interpreted like in IQC-frames except

Definition

$$
\begin{array}{lll}
\mathfrak{M} \models^{\sigma}_{w} B \to C & \text{iff} & \text{if } \mathfrak{M} \models^{\sigma}_{w} B, \text{ then } \mathfrak{M} \models^{\sigma}_{w} C \\
\mathfrak{M} \models^{\sigma}_{w} \forall x B & \text{iff} & \text{for each } w\text{-assignment } \tau \\
 & & \text{that is an } x\text{-variant of } \sigma, \mathfrak{M} \models^{\tau}_{w} B \\
\mathfrak{M} \models^{\sigma}_{w} \Box B & \text{iff} & \text{for all } v \text{ with } wRv, \mathfrak{M} \models^{\sigma}_{v} B
\end{array}
$$

The definitions of truth in a model and validity in a frame are like in IQC.

Theorem (Gabbay 1976)

QK *is sound and complete with respect to the class of predicate Kripke frames; that is, for each formula* A

$$\mathsf{QK} \vdash A \quad \textit{iff} \quad \mathfrak{F} \models A \textit{ for each predicate Kripke frame } \mathfrak{F}.$$

Kripke semantics for QS4

Definition

A QS4-frame is a QK-frame in which the relation R is reflexive and transitive (quasi-order).

Theorem (Hughes-Cresswell (1968), Schütte (1968))

QS4 *is sound and complete with respect to the class of* QS4 *frames; that is, for each formula* A

$$\mathsf{QS4} \vdash A \quad \textit{iff} \quad \mathfrak{F} \models A \textit{ for each } \mathsf{QS4}\textit{-frame } \mathfrak{F}.$$

Barcan and converse Barcan formulas

$\Box\forall x A \to \forall x \Box A$	converse Barcan formula	(CBF)
$\forall x \Box A \to \Box\forall x A$	Barcan formula	(BF)

Barcan and converse Barcan formulas

$\Box\forall xA \to \forall x\Box A$	converse Barcan formula	(CBF)
$\forall x\Box A \to \Box\forall xA$	Barcan formula	(BF)

Proposition

- QK $\vdash$ CBF

Barcan and converse Barcan formulas

$\Box\forall x A \to \forall x \Box A$	converse Barcan formula	(CBF)
$\forall x \Box A \to \Box\forall x A$	Barcan formula	(BF)

Proposition

- QK $\vdash$ CBF

QK $\vdash$ CBF

1. $\forall x A \to A$
2. $\Box(\forall x A \to A)$
3. $\Box\forall x A \to \Box A$
4. $\forall x(\Box\forall x A \to \Box A)$
5. $\Box\forall x A \to \forall x \Box A$

Barcan and converse Barcan formulas

$\Box\forall x A \to \forall x \Box A$	converse Barcan formula	(CBF)
$\forall x \Box A \to \Box\forall x A$	Barcan formula	(BF)

Proposition

- $\mathrm{QK} \vdash \mathrm{CBF}$
- $\mathfrak{F} \models \mathrm{BF}$ iff $\mathfrak{F}$ has constant domains, i.e. $wRv \;\Rightarrow D_w = D_v$.
- $\mathrm{QK} \nvdash \mathrm{BF}$
- $\mathrm{QK} + \mathrm{BF}$ is complete with respect to the class of predicate Kripke frames with constant domains (Gabbay 1976)

QS4.t

Let $\mathcal{L}_T$ be a predicate bimodal language with two modalities $\Box_F$ and $\Box_P$.

Definition

QS4.t is the smallest set of formulas of $\mathcal{L}_T$ containing all the substitution instances of the S4.t-theorems, the axiom schemes

1. $\forall x A \to A(y/x)$ Universal instantiation (UI)
2. $\forall x(A \to B) \to (A \to \forall x B)$ with x not free in A

and closed under (MP), (Gen) and

$$\frac{A}{\Box_F A} \quad \Box_F\text{-Necessitation } (\mathrm{N_F}) \qquad\qquad \frac{A}{\Box_P A} \quad \Box_P\text{-Necessitation } (\mathrm{N_P})$$

QS4.t-frames

Definition

QS4.t-frames are QS4-frames with constant domains. We interpret the temporal modalities as follows

$$
\begin{array}{lll}
\mathfrak{M} \models^{\sigma}_{w} \Box_F B & \text{iff} & (\forall v \in W)(wRv \Rightarrow \mathfrak{M} \models^{\sigma}_{v} B) \\
\mathfrak{M} \models^{\sigma}_{w} \Box_P B & \text{iff} & (\forall v \in W)(vRw \Rightarrow \mathfrak{M} \models^{\sigma}_{v} B)
\end{array}
$$

Theorem

QS4.t *is sound and complete with respect to the class of* QS4.t-*frames.*

Barcan and converse Barcan formulas for tense logics

$\Box_F \forall x A \to \forall x \Box_F A$	converse Barcan formula for $\Box_F$	$(\mathrm{CBF_F})$
$\forall x \Box_F A \to \Box_F \forall x A$	Barcan formula for $\Box_F$	$(\mathrm{BF_F})$
$\Box_P \forall x A \to \forall x \Box_P A$	converse Barcan formula for $\Box_P$	$(\mathrm{CBF_P})$
$\forall x \Box_P A \to \Box_P \forall x A$	Barcan formula for $\Box_P$	$(\mathrm{BF_P})$

Barcan and converse Barcan formulas for tense logics

$\Box_F \forall x A \to \forall x \Box_F A$	converse Barcan formula for $\Box_F$	($\mathrm{CBF_F}$)
$\forall x \Box_F A \to \Box_F \forall x A$	Barcan formula for $\Box_F$	($\mathrm{BF_F}$)
$\Box_P \forall x A \to \forall x \Box_P A$	converse Barcan formula for $\Box_P$	($\mathrm{CBF_P}$)
$\forall x \Box_P A \to \Box_P \forall x A$	Barcan formula for $\Box_P$	($\mathrm{BF_P}$)

Proposition

- $\mathrm{QS4.t} \vdash \mathrm{CBF_F}, \mathrm{CBF_P}$

Barcan and converse Barcan formulas for tense logics

$\Box_F \forall x A \to \forall x \Box_F A$	converse Barcan formula for $\Box_F$	$(\mathrm{CBF_F})$
$\forall x \Box_F A \to \Box_F \forall x A$	Barcan formula for $\Box_F$	$(\mathrm{BF_F})$
$\Box_P \forall x A \to \forall x \Box_P A$	converse Barcan formula for $\Box_P$	$(\mathrm{CBF_P})$
$\forall x \Box_P A \to \Box_P \forall x A$	Barcan formula for $\Box_P$	$(\mathrm{BF_P})$

Proposition

- $\mathrm{QS4.t} \vdash \mathrm{CBF_F}, \mathrm{CBF_P}$

$\mathrm{QS4.t} \vdash \mathrm{CBF_F}$

1. $\forall x A \to A$
2. $\Box_F(\forall x A \to A)$
3. $\Box_F \forall x A \to \Box_F A$
4. $\forall x(\Box_F \forall x A \to \Box_F A)$
5. $\Box_F \forall x A \to \forall x \Box_F A$

$\mathrm{QS4.t} \vdash \mathrm{CBF_P}$

1. $\forall x A \to A$
2. $\Box_P(\forall x A \to A)$
3. $\Box_P \forall x A \to \Box A$
4. $\forall x(\Box_P \forall x A \to \Box_P A)$
5. $\Box_P \forall x A \to \forall x \Box_P A$

Barcan and converse Barcan formulas for tense logics

$\Box_F \forall x A \to \forall x \Box_F A$	converse Barcan formula for $\Box_F$	$(\mathrm{CBF_F})$
$\forall x \Box_F A \to \Box_F \forall x A$	Barcan formula for $\Box_F$	$(\mathrm{BF_F})$
$\Box_P \forall x A \to \forall x \Box_P A$	converse Barcan formula for $\Box_P$	$(\mathrm{CBF_P})$
$\forall x \Box_P A \to \Box_P \forall x A$	Barcan formula for $\Box_P$	$(\mathrm{BF_P})$

Proposition

- $\mathrm{QS4.t} \vdash \mathrm{CBF_F}, \mathrm{CBF_P}$
- $\mathrm{QS4.t} \vdash \mathrm{BF_F}, \mathrm{BF_P}$

Barcan and converse Barcan formulas for tense logics

$\mathsf{QS4.t} \vdash \mathsf{BF_F}$

1. $\forall x B \to B$
2. $\Box_P(\forall x B \to B)$
3. $\Box_P(\forall x B \to B) \to (\Diamond_P \forall x B \to \Diamond_P B)$
4. $\Diamond_P \forall x B \to \Diamond_P B$
5. $\forall x(\Diamond_P \forall x B \to \forall x \Diamond_P B)$
6. $\Diamond_P \forall x B \to \forall x \Diamond_P B$
7. $\forall x \Box_F A \to \Box_F \Diamond_P \forall x \Box_F A$
8. $\Diamond_P \forall x \Box_F A \to \forall x \Diamond_P \Box_F A$
9. $\Box_F \Diamond_P \forall x \Box_F A \to \Box_F \forall x \Diamond_P \Box_F A$
10. $\Diamond_P \Box_F A \to A$
11. $\forall x \Diamond_P \Box_F A \to \forall x A$
12. $\Box_F \forall x \Diamond_P \Box_F A \to \Box_F \forall x A$
13. $\forall x \Box_F A \to \Box_F \forall x A$

Barcan and converse Barcan formulas for tense logics

$\mathrm{QS4.t} \vdash \mathrm{BF_P}$

1. $\forall xB \to B$
2. $\Box_F(\forall xB \to B)$
3. $\Box_F(\forall xB \to B) \to (\Diamond_F \forall xB \to \Diamond_F B)$
4. $\Diamond_F \forall xB \to \Diamond_F B$
5. $\forall x(\Diamond_F \forall xB \to \forall x \Diamond_F B)$
6. $\Diamond_F \forall xB \to \forall x \Diamond_F B$
7. $\forall x \Box_P A \to \Box_P \Diamond_F \forall x \Box_P A$
8. $\Diamond_F \forall x \Box_P A \to \forall x \Diamond_F \Box_P A$
9. $\Box_P \Diamond_F \forall x \Box_P A \to \Box_P \forall x \Diamond_F \Box_P A$
10. $\Diamond_F \Box_P A \to A$
11. $\forall x \Diamond_F \Box_P A \to \forall xA$
12. $\Box_P \forall x \Diamond_F \Box_P A \to \Box_P \forall xA$
13. $\forall x \Box_P A \to \Box_P \forall xA$

QS4.t cannot be the target

$$\forall x(A \vee B) \to (A \vee \forall x B) \quad \text{with } x \text{ not free in } A \tag{CD}$$

Proposition

- Let $\mathfrak{F}$ be an IQC-frame. $\mathfrak{F} \vdash \mathrm{CD}$ iff $\mathfrak{F}$ has constant domains.
- $\mathrm{IQC} \nvdash \mathrm{CD}$
- $\mathrm{QS4.t} \vdash (\mathrm{CD})^t$

Therefore, QS4.t is **not** the right candidate to be the target of our temporal translation.

Weakening the universal instantiation axiom

We are looking for a tense predicate logic that does not prove $\mathsf{BF_F}$ and $\mathsf{CBF_P}$ because we do not want constant domains.

Notice that the QS4.t-proofs of $\mathsf{CBF_F}, \mathsf{CBF_P}, \mathsf{BF_F}, \mathsf{BF_P}$ all use the universal instantiation axiom. So we replace the universal instantiation axiom by its weaker version

$$\forall y(\forall x A \to A(y/x))$$

History

- Kripke (1963) was the first to considered the weak universal instantiation axiom. His goal was to have a predicate modal logic that did not prove either CBF nor BF. He also gave a semantics for this logic. In these frames the variables are interpreted in the the union of all the domains. He did not prove completeness.
- Hughes and Cresswell (1996) introduced a similar predicate modal logic and proved its completeness with respect to a generalized Kripke semantics.
- Fitting and Mendelsohn (1998) gave an alternate axiomatization of this logic.
- Corsi (2002) defined the system $\mathrm{Q}^{\circ}\mathrm{K}$. She proved completeness with respect to a generalized Kripke semantics. Each world of a frame has two associated domains, an inner and an outer one. She also proved completeness of $\mathrm{Q}^{\circ}\mathrm{K} + \mathrm{CBF}$ and $\mathrm{Q}^{\circ}\mathrm{K} + \mathrm{CBF} + \mathrm{BF}$.

Q°K

Definition

The logic Q°K is the least set of formulas of $\mathcal{L}_\Box$ containing all the substitution instances of K-theorems, the axiom schemes

1. $\forall y(\forall x A \to A(y/x))$ (UI°)
2. $\forall x(A \to B) \to (\forall x A \to \forall x B)$
3. $\forall x \forall y A \leftrightarrow \forall y \forall x A$
4. $A \to \forall x A$ with x not free in A

and closed under (MP), (Gen), and (N).

Remark

Replacing (UI°) with (UI) gives an axiomatization of QK.

Generalized Kripke semantics

Definition

A generalized Kripke frame is a quadruple $\mathfrak{F} = (W, R, D, U)$ where

- W is a nonempty set whose elements are called the *worlds* of $\mathfrak{F}$.
- R is a binary relation on W.
- D is a function that associates to each $w \in W$ a set D_w. The set D_w is called the *inner domain* of w.
- U is a function that associates to each $w \in W$ a nonempty set U_w such that $D_w \subseteq U_w$ and wRv implies $U_w \subseteq U_v$. The set U_w is called the *outer domain* of w.

Generalized Kripke semantics

Definition

- An *interpretation* of $\mathcal{L}_\Box$ in $\mathfrak{F}$ is a function I associating to each world w and an n-ary predicate symbol P an n-ary relation $I_w(P) \subseteq (U_w)^n$.
- A *model* is a pair $\mathfrak{M} = (\mathfrak{F}, I)$ where $\mathfrak{F}$ is a frame and I is an interpretation in $\mathfrak{F}$.
- A *w-assignment* in $\mathfrak{F}$ is a function σ that associates to each individual variable an element of U_w.
- If σ and τ are two w-assignments and x is an individual variable, τ is said to be an *x-variant* of σ if $\tau(y) = \sigma(y)$ for all $y \neq x$.
- We say that a w-assignment σ is *w-inner* for $w \in W$ if $\sigma(x) \in D_w$ for each individual variable x.

Generalized Kripke semantics

The connectives are interpreted like in QK-frames and

Definition

$$
\begin{array}{lll}
\mathfrak{M} \models_w^\sigma \exists x B & \text{iff} & \text{for some } x\text{-variant } \tau \text{ of } \sigma \\
 & & \text{with } \tau(x) \in D_w,\ \mathfrak{M} \models_w^\tau B \\
\mathfrak{M} \models_w^\sigma \forall x B & \text{iff} & \text{for all } x\text{-variants } \tau \text{ of } \sigma \\
 & & \text{with } \tau(x) \in D_w,\ \mathfrak{M} \models_w^\tau B
\end{array}
$$

The definitions of truth in a model and validity in a frame coincide with the ones for QK-frames.

Theorem (Corsi 2002)

$Q^\circ K$ *is sound and complete with respect to this semantics.*

CBF, BF, NID and UI in Q°K-frames

$\Box\forall x A \rightarrow \forall x \Box A$	(CBF)	increasing inner domains	$wRv \;\Rightarrow\; D_w \subseteq D_v$
$\forall x \Box A \rightarrow \Box\forall x A$	(BF)	decreasing inner domains	$wRv \;\Rightarrow\; D_v \subseteq D_w$
$\forall x A \rightarrow A$ with x not free in A	(NID)	nonempty inner domains	$D_w \neq \emptyset$
$\forall x A \rightarrow A(y/x)$	(UI)	inner=outer	$D_w = U_w$

Theorem (Corsi 2002)

Q°K + NID, Q°K + CBF(+NID), *and* Q°K + CBF + BF(+NID) *are sound and complete with respect to the relative classes of generalized frames.*

Completeness of Q°K + BF is an open problem.

$Q^{\circ}S4.t$

Definition

The logic $Q^{\circ}S4.t$ is the least set of formulas of $\mathcal{L}_T$ containing all the substitution instances of the S4.t-axioms, the axiom schemes

1. $\forall y(\forall xA \to A(y/x))$ $\quad$ (UI$^{\circ}$)
2. $\forall x(A \to B) \to (\forall xA \to \forall xB)$
3. $\forall x\forall yA \leftrightarrow \forall y\forall xA$
4. $A \to \forall xA$ with x not free in A
5. $\forall xA \to A$ with x not free in A $\quad$ (NID)
6. $\Box_F\forall xA \to \forall x\Box_F A$ $\quad$ ($\mathrm{CBF_F}$)

and closed under (MP), (Gen), ($\mathrm{N_F}$), and ($\mathrm{N_P}$).

Generalized Kripke semantics for Q°S4.t

Definition

A Q°S4.t-*frame* is a generalized Kripke frame $\mathfrak{F} = (W, R, D, U)$ such that

- R is a quasi-order on W.
- The inner domains are nonempty and increasing
- U_w is the same for all $w \in W$. We denote it with U and we call it the *outer domain* of $\mathfrak{F}$.

Interpretations, models, assignments are the defined like for Q°K. Since the outer domain is the same for each world, we say assignments instead of w-assignments.

Generalized Kripke semantics for Q°S4.t

We interpret the temporal modalities in the standard way, the other connectives and quantifiers are interpreted like in Q°K-frames.

Definition

$$
\begin{array}{lcl}
\mathfrak{M} \models^{\sigma}_{w} \Box_F B & \text{iff} & (\forall v \in W)(wRv \Rightarrow \mathfrak{M} \models^{\sigma}_{v} B) \\
\mathfrak{M} \models^{\sigma}_{w} \Box_P B & \text{iff} & (\forall v \in W)(vRw \Rightarrow \mathfrak{M} \models^{\sigma}_{v} B)
\end{array}
$$

The definitions of truth in a model and validity coincide with the ones for QK-frames.

Theorem

Q°S4.t *is sound with respect to the class of* Q°S4.t*-frames; that is, for each formula A*

$$\textit{if } \ \mathrm{Q^{\circ}S4.t} \vdash A \ \textit{ then } \ \mathfrak{F} \models A \textit{ for each } \mathrm{Q^{\circ}S4.t}\textit{-frame } \mathfrak{F}.$$

Completeness is still an open problem.

Problem with faithfulness

It is not true in general that

$$\mathsf{IQC} \vdash A \quad \Rightarrow \quad \mathsf{Q^\circ S4.t} \vdash A^t$$

for example when A is the universal instantiation axiom. Thus, the translation is not faithful in the standard sense.

Main theorem

Theorem

- *For any formula A in $\mathcal{L}$, we have*

$$\mathrm{IQC} \vdash A \quad \textit{iff} \quad \mathrm{Q^\circ S4.t} \vdash \forall x_1 \cdots \forall x_n A^t$$

  where $x_1, \ldots, x_n$ are the free variables in A.

- *If A is a sentence, then*

$$\mathrm{IQC} \vdash A \quad \textit{iff} \quad \mathrm{Q^\circ S4.t} \vdash A^t.$$

If A contains constants, they first need to be replaced with fresh variables.

Faithfulness

$$\mathsf{IQC} \vdash A \quad \Rightarrow \quad \mathsf{Q}^{\circ}\mathsf{S4.t} \vdash \forall x_1 \cdots \forall x_n A^t$$

Faithfulness is proved syntactically by induction on the length of the IQC-proof of A.

Proof of faithfulness

$$(\forall x A \to A(y/x))^t \quad = \Box_F(\Box_F \forall x A^t \to A(y/x)^t)$$

$$(A(y/x) \to \exists x A)^t \quad = \Box_F(A(y/x)^t \to \Diamond_P \exists x A^t)$$

$$(\forall x(A \to B) \to (A \to \forall x B))^t$$
$$= \Box_F(\Box_F \forall x \Box_F(A^t \to B^t) \to \Box_F(A^t \to \Box_F \forall x B^t))$$

$$(\forall x(A \to B) \to (\exists x A \to B))^t$$
$$= \Box_F(\Box_F \forall x \Box_F(A^t \to B^t) \to \Box_F(\Diamond_P \exists x A^t \to B^t))$$

Proof of faithfulness

Lemma

If A is an instance of an axiom scheme of IQC *and* $\mathbf{x}$ *is the list of free variables in A, then* $\mathrm{Q}^{\circ}\mathrm{S4.t} \vdash \forall \mathbf{x}\, A^t$.

Lemma

Let A, B be formulas of $\mathcal{L}$, $\mathbf{x}$ *the list of variables free in $A \to B$,* $\mathbf{y}$ *the list of variables free in A, and* $\mathbf{z}$ *the list of variables free in B. If* $\mathrm{Q}^{\circ}\mathrm{S4.t} \vdash \forall \mathbf{x}(A \to B)^t$ *and* $\mathrm{Q}^{\circ}\mathrm{S4.t} \vdash \forall \mathbf{y} A^t$, *then* $\mathrm{Q}^{\circ}\mathrm{S4.t} \vdash \forall \mathbf{z} B^t$.

Lemma

Let A be a formula of $\mathcal{L}$, x a variable, $\mathbf{y}$ *the list of variables free in A, and* $\mathbf{z}$ *the list of variables free in $\forall x A$. If* $\mathrm{Q}^{\circ}\mathrm{S4.t} \vdash \forall \mathbf{y} A^t$, *then* $\mathrm{Q}^{\circ}\mathrm{S4.t} \vdash \forall \mathbf{z}\, (\forall x A)^t$.

Fullness

$$\mathsf{IQC} \nvdash A \quad \Rightarrow \quad \mathsf{Q}^{\circ}\mathsf{S4.t} \nvdash \forall x_1 \cdots \forall x_n A^t$$

To prove fullness we use semantical methods. The strategy is to show that to any IQC-model $\mathfrak{M}$ can be associated a Q$^{\circ}$S4.t-model $\overline{\mathfrak{M}}$ such that if A is refuted in $\mathfrak{M}$ then $\forall x_1 \cdots \forall x_n A^t$ is refuted in $\overline{\mathfrak{M}}$.

Relation between IQC-models and Q°S4.t-models

Definition

- For an IQC-frame $\mathfrak{F} = (W, R, D)$ let $\overline{\mathfrak{F}} = (W, R, D, U)$ where $U = \bigcup\{D_w \mid w \in W\}$.
- For an IQC-model $\mathfrak{M} = (\mathfrak{F}, I)$ let $\overline{\mathfrak{M}} = (\overline{\mathfrak{F}}, I)$.

Remark

- It is obvious that $\overline{\mathfrak{F}}$ is a Q°S4.t-frame.
- If I is an interpretation in $\mathfrak{F}$, then I is also an interpretation in $\overline{\mathfrak{F}}$ because for each n-ary predicate letter P we have $I_w(P) \subseteq D_w^n \subseteq U^n$. Therefore, $\overline{\mathfrak{M}}$ is well defined.
- The w-assignments in $\mathfrak{F}$ are exactly the w-inner assignments in $\overline{\mathfrak{F}}$.

Sketch proof of fullness

Lemma

- *If A is a formula of $\mathcal{L}$, then* $\mathrm{Q}^\circ\mathrm{S4.t} \vdash A^t \to \Box_F A^t$.
- *Therefore, if* $\mathfrak{N}$ *is a* $\mathrm{Q}^\circ\mathrm{S4.t}$*-model,* σ *an assignment and* wRv*, then* $\mathfrak{N} \models^\sigma_w A^t$ *implies* $\mathfrak{N} \models^\sigma_v A^t$.

Proposition

Let A be a formula of $\mathcal{L}$, $\mathfrak{M} = (\mathfrak{F}, I)$ an IQC-model based on an IQC-frame $\mathfrak{F} = (W, R, D)$, and $w \in W$.

- For each w-assignment σ,

$$\mathfrak{M} \models^\sigma_w A \ \text{ iff } \ \overline{\mathfrak{M}} \models^\sigma_w A^t.$$

- If $x_1, \ldots, x_n$ are the free variables of A, then

$$\mathfrak{M} \models_w A \ \text{ iff } \ \overline{\mathfrak{M}} \models_w \forall x_1 \cdots \forall x_n A^t.$$

Sketch proof of fullness

If $A = \exists x B$, then

$$\mathfrak{M} \models^{\sigma}_{w} \exists x B \text{ iff there is a } w\text{-assignment } \tau \text{ that is an } x\text{-variant of } \sigma$$
$$\text{such that } \mathfrak{M} \models^{\tau}_{w} B$$

Sketch proof of fullness

If $A = \exists x B$, then

$$\begin{aligned}
\mathfrak{M} \vDash^{\sigma}_{w} \exists x B \text{ iff } & \text{there is a } w\text{-assignment } \tau \text{ that is an } x\text{-variant of } \sigma \\
& \quad \text{such that } \mathfrak{M} \vDash^{\tau}_{w} B \\
\text{iff } & \text{there is an assignment } \tau \text{ that is an } x\text{-variant of } \sigma \\
& \quad \text{with } \tau(x) \in D_w \text{ such that } \overline{\mathfrak{M}} \vDash^{\tau}_{w} B^t
\end{aligned}$$

By induction hypothesis and the correspondence between assignments on $\mathfrak{F}$ and on $\overline{\mathfrak{F}}$.

Sketch proof of fullness

If $A = \exists x B$, then

$$\begin{aligned}
\mathfrak{M} \vDash^{\sigma}_{w} \exists x B \ &\text{iff there is a } w\text{-assignment } \tau \text{ that is an } x\text{-variant of } \sigma \\
&\quad \text{such that } \mathfrak{M} \vDash^{\tau}_{w} B \\
&\text{iff there is an assignment } \tau \text{ that is an } x\text{-variant of } \sigma \\
&\quad \text{with } \tau(x) \in D_w \text{ such that } \overline{\mathfrak{M}} \vDash^{\tau}_{w} B^t \\
&\text{iff there is } v \in W \text{ such that } vRw \text{ and an assignment } \rho \text{ that is} \\
&\quad \text{an } x\text{-variant of } \sigma \text{ with } \rho(x) \in D_v \text{ such that } \overline{\mathfrak{M}} \vDash^{\rho}_{v} B^t \\
&\text{iff } \overline{\mathfrak{M}} \vDash^{\sigma}_{w} \Diamond_P \exists x B^t \\
&\text{iff } \overline{\mathfrak{M}} \vDash^{\sigma}_{w} (\exists x B)^t
\end{aligned}$$

By reflexivity of R, the Lemma above, and the fact that vRw implies $D_v \subseteq D_w$.

Open problems and future directions

- Completeness of $Q^{\circ}S4.t$.
- Study of logics with weak universal instantiation axiom.
- Extending this result to intermediate logics
- Can $Q^{\circ}S4.t$ be replaced by other logics?

Thanks for your attention!