Modal operators on rings of continuous functions

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joint work with

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Nonclassical Logic Webinar

University of Denver

November 20, 2020

Introduction

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- Stone duality can be extended to a dual equivalence between the categories modal algebras and descriptive frames, i.e. Stone spaces with a continuous binary relation. Such a duality is called Jónsson-Tarski duality (full duality is due to Halmos, Esakia, and Goldblatt).

$$\boldsymbol{BA} \longleftrightarrow \boldsymbol{Stone} \qquad\qquad \boldsymbol{MA} \longleftrightarrow \boldsymbol{DF}$$

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KHaus

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- Gelfand-Naimark-Stone duality establishes a dual equivalence between the category of uniformly complete bounded archimedean ℓ-algebras and the category of compact Hausdorff spaces.

$$\textbf{\textit{uba}}\boldsymbol{\ell} \longleftrightarrow \textbf{\textit{KHaus}}$$

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- We want to define modal operators on bounded archimedean ℓ-algebras in order to obtain a dual equivalence with compact Hausdorff frames.

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Outline

1. Gelfand duality
2. Modal extension of Gelfand duality
3. Duality via algebras/coalgebras
4. Consequences

Table of Contents

Gelfand duality

- Gelfand and Naimark (1943): the category ***KHaus*** of compact Hausdorff spaces is dually equivalent to the category of commutative C^*-algebras.

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- The two dualities are strictly related: complexification of Stone rings are commutative C^*-algebras. On the other hand, self-adjoint elements of commutative C^*-algebras form Stone rings.
- Similar dualities were investigated by the Krein brothers, Kakutani, and Yosida (vector lattices or Riesz spaces) and later by Henriksen and Johnson.

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- Many rings of real-valued functions are examples of bounded archimedean ℓ-algebras (continuous, piecewise constant, and piecewise polynomial). Stone rings correspond to the uniformly complete bounded archimedean ℓ-algebras.

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- A is an $\mathbb{R}$-algebra,
- $0 \leq a \in A$ and $0 \leq \lambda \in \mathbb{R}$ imply $0 \leq \lambda \cdot a$.

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- Let $\boldsymbol{ba\ell}$ be the category of bounded archimedean ℓ-algebras and unital ℓ-algebra homomorphisms.

From ***KHaus*** to ***baℓ***: ring of continuous functions $C(X)$

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  If $g \in C(Y)$, then $C(f)(g) := g \circ f \in C(X)$.

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The rings of piecewise constant and piecewise polynomial functions on form two bounded archimedean ℓ-algebras that are not usually uniformly complete.

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Proposition

$C(X)$ is uniformly complete for each X compact Hausdorff.

Adjunction and duality

Lemma (Stone)

Let $X \in KHaus$.

- *If $x \in X$, then $\{f \in C(X) \mid f(x) = 0\}$ is a maximal ℓ-ideal of $C(X)$.*

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Let $X \in KHaus$.

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- *The map $\varepsilon_X : X \to Y_{C(X)}$ defined by $\varepsilon_X(x) = \{f \in C(X) \mid f(x) = 0\}$ is a homeomorphism.*

Adjunction and duality: ζ_A

Lemma

Let $A \in \boldsymbol{ba\ell}$.

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- $\zeta_A(a) : Y_A \to \mathbb{R}$ *is a continuous function for each* $a \in A$.

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- *$\zeta_A(a) : Y_A \to \mathbb{R}$ is a continuous function for each $a \in A$.*
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- *ζ_A embeds A into the uniformly complete $C(Y_A)$.*

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- *ζ_A embeds A into the uniformly complete $C(Y_A)$.*
- *Since $\zeta_A(A)$ separates the points of Y_A, it is a uniformly dense subalgebra of $C(Y_A)$ by the Stone-Weierstrass Theorem.*

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- *If A is uniformly complete, then ζ_A is an isomorphism.*

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- *$\zeta_A(a) : Y_A \to \mathbb{R}$ is a continuous function for each $a \in A$.*
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Theorem (Representation)

*Each $A \in \textbf{\textit{ba}}\boldsymbol{\ell}$ is isomorphic to a uniformly dense subalgebra of $C(X)$ for some $X \in$ **KHaus**.*

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Lemma

- $\varepsilon : Id_{\mathbf{KHaus}} \to YC$ *is a natural isomorphism.*

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Theorem

There is a dual adjunction between $\mathbf{ba\ell}$ *and* $\mathbf{KHaus}$ *whose unit and counit are* ε *and* ζ*.*

$$\mathbf{ba\ell} \underset{Y}{\overset{C}{\leftrightarrows}} \mathbf{KHaus}$$

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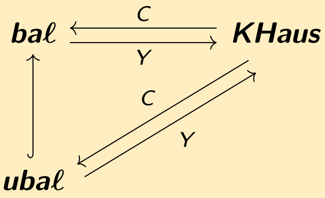
There is a dual adjunction between $\boldsymbol{ba\ell}$ *and* $\boldsymbol{KHaus}$ *whose unit and counit are* ε *and* ζ. *This adjunction restricts to a dual equivalence between* $\boldsymbol{uba\ell}$ *and* $\boldsymbol{KHaus}$.



Adjunction and duality

Lemma

- $\varepsilon : Id_{\mathbf{KHaus}} \to YC$ *is a natural isomorphism.*
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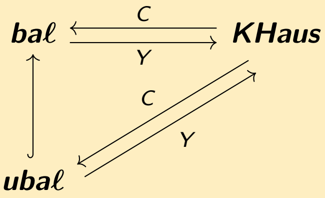

ubaℓ *is a reflective subcategory of* ***baℓ*** *and* $CY : \mathbf{ba\ell} \to \mathbf{uba\ell}$ *is a reflector.*

Table of Contents

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- A map $f : (X, R) \to (Y, S)$ between compact Hausdorff frames is a p-morphism if $f(R[x]) = S[f(x)]$ for each $x \in X$.
- We denote the category of compact Hausdorff frames and continuous p-morphisms with ***KHF***.

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Definition

Let $f \in C(X)$. We define

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Thus, $\Box_R f$ is continuous.

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- $(\Box_R f)(x) = \inf f(R[x])$,
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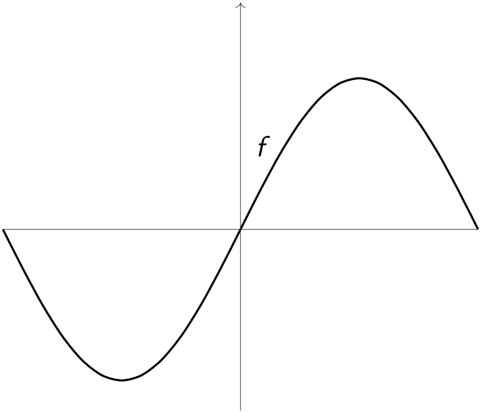
From continuous relations to modal operators

If R is a partial order on X and $f \in C(X)$, then



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If R is a partial order on X and $f \in C(X)$, then
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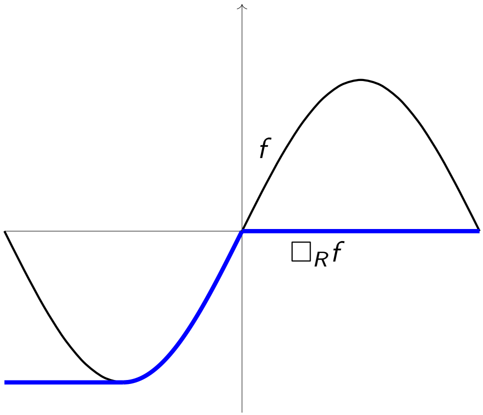

From continuous relations to modal operators

If R is a partial order on X and $f \in C(X)$, then
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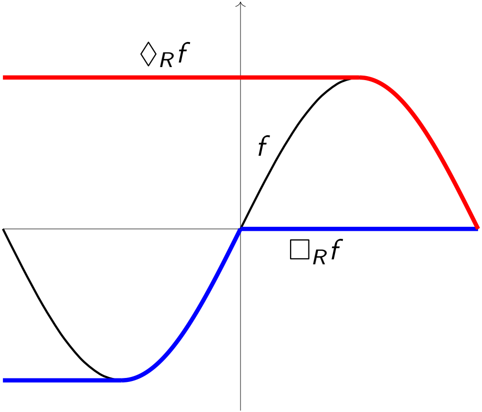

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5. If $0 \leq \lambda$, then $\Box_R(\lambda f) = \lambda \Box_R f$.

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Modal bounded archimedean ℓ-algebras

Definition

- Let $A \in \boldsymbol{ba\ell}$. We say that a unary function $\Box : A \to A$ is a modal operator on A provided $\Box$ satisfies the following axioms for each $a, b \in A$ and $\lambda \in \mathbb{R}$:

(M1) $\Box(a \wedge b) = \Box a \wedge \Box b$.
(M2) $\Box\lambda = \lambda + (1 - \lambda)\Box 0$.
(M3) $\Box(a^+) = (\Box a)^+$ (where $a^+ = a \vee 0$).
(M4) $\Box(a + \lambda) = \Box a + \Box\lambda - \Box 0$.
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- If $\Box$ is a modal operator on $A \in \boldsymbol{ba\ell}$, then we call the pair $(A, \Box)$ a modal bounded archimedean ℓ-algebra.

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- If $\Box$ is a modal operator on $A \in \boldsymbol{ba\ell}$, then we call the pair $(A, \Box)$ a modal bounded archimedean ℓ-algebra.
- Let $\boldsymbol{mba\ell}$ be the category of modal bounded archimedean ℓ-algebras and unital ℓ-algebra homomorphisms preserving $\Box$.
- Let $\boldsymbol{muba\ell}$ be the full subcategory of uniformly complete objects of $\boldsymbol{mba\ell}$.

From modal operators to continuous relations

Lemma

Let $(X, R) \in \textbf{\textit{KHF}}$ and $x, y \in X$. Then

$$xRy \text{ iff for each } f \geq 0,\ f(y) = 0, \text{ implies } (\Box_R f)(x) = 0.$$

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Also, $\zeta_A(a) \in C(Y_A)$ vanishes exactly on the y such that $a \in y$.
Indeed, $a \in y$ iff $\zeta_A(a)(y) = a + y = 0 + y$.

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This suggests the following definition of $R_\Box$ on Y_A.

Definition

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$$a \geq 0,\ a \in y \ \text{ implies } \ \Box a \in x$$

From modal operators to continuous relations

The family

$$Z_\ell(a) = \{x \in Y_A \mid a \in x\} \text{ where } a \in A,\ a \geq 0$$

forms a basis of closed sets of Y_A.

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- $R_\Box^{-1}[Z_\ell(a)] = Z_\ell(\Box a)$.

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Lemma

Let $(A, \Box) \in \boldsymbol{mba\ell}$ and $a \in A$, $a \geq 0$.

- *$R_\Box[x]$ is closed for each $x \in Y_A$.*
- *$R_\Box^{-1}[Z_\ell(a)] = Z_\ell(\Box a)$.*
- *$R_\Box^{-1}[Y_A \setminus Z_\ell(a)] = Y_A \setminus Z_\ell(\Diamond a)$.*

From modal operators to continuous relations

Lemma (Esakia Lemma)

Let $(X, R) \in \boldsymbol{KHF}$*. Let* $\mathcal{F}$ *be a nonempty downward directed family of closed subsets of* X *(i.e.* $\forall A, B \in \mathcal{F}$, $\exists C \in \mathcal{F}$ *such that* $C \subseteq A \cap B$*). Then*

$$R^{-1} \bigcap \{F \mid F \in \mathcal{F}\} = \bigcap \{R^{-1}[F] \mid F \in \mathcal{F}\}$$

From modal operators to continuous relations

Lemma (Esakia Lemma)

*Let $(X, R) \in$ **KHF**. Let $\mathcal{F}$ be a nonempty downward directed family of closed subsets of X (i.e. $\forall A, B \in \mathcal{F}$, $\exists C \in \mathcal{F}$ such that $C \subseteq A \cap B$). Then*

$$R^{-1} \bigcap \{F \mid F \in \mathcal{F}\} = \bigcap \{R^{-1}[F] \mid F \in \mathcal{F}\}$$

Since every closed subset of Y_A is intersection of a downward directed family of sets of the form $Z_\ell(a)$ with $a \geq 0$, the previous lemma yields

From modal operators to continuous relations

Lemma (Esakia Lemma)

Let $(X, R) \in \boldsymbol{KHF}$. Let $\mathcal{F}$ be a nonempty downward directed family of closed subsets of X (i.e. $\forall A, B \in \mathcal{F}$, $\exists C \in \mathcal{F}$ such that $C \subseteq A \cap B$). Then

$$R^{-1} \bigcap \{F \mid F \in \mathcal{F}\} = \bigcap \{R^{-1}[F] \mid F \in \mathcal{F}\}$$

Since every closed subset of Y_A is intersection of a downward directed family of sets of the form $Z_\ell(a)$ with $a \geq 0$, the previous lemma yields

Theorem

$R_\Box$ is a continuous relation on Y_A.

Adjunction and duality

Theorem

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Let $(A,\Box) \in \boldsymbol{mba\ell}$ and $(X,R) \in \boldsymbol{KHF}$.

- *for each $x, y \in X$ we have xRy iff $\varepsilon_X(x) R_{\Box_R} \varepsilon_X(y)$
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- *$\zeta : Id_{\textbf{mba}\ell} \to CY$ is a natural transformation.*

Adjunction and duality

Theorem (Main theorem)

There is a dual adjunction between $\boldsymbol{mba\ell}$ and $\boldsymbol{KHF}$ whose unit and counit are ε and ζ.

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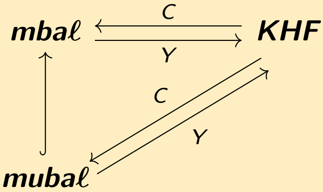

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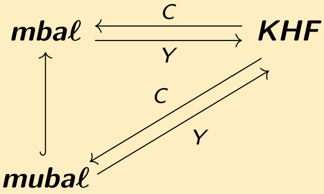

$\boldsymbol{muba\ell}$ *is a reflective subcategory of* $\boldsymbol{mba\ell}$ *and* $CY : \boldsymbol{mba\ell} \to \boldsymbol{muba\ell}$ *is a reflector.*

Table of Contents

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- Let $\boldsymbol{Coalg}(\mathcal{T})$ be the category whose objects are coalgebras for $\mathcal{T}$ and whose morphisms are morphisms of coalgebras.

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The definition of algebras for an endofunctor is dual.

Vietoris space

Definition

Let $X \in \boldsymbol{KHaus}$ and $\mathcal{V}(X)$ be the set of its closed subsets.
If U is an open subset of X consider the following subsets of $\mathcal{V}(X)$.

$$\Box_U = \{F \in \mathcal{V}(X) \mid F \subseteq U\},$$
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Moreover, $\mathcal{V}$ is an endofunctor on $\textbf{\textit{KHaus}}$.

Coalgebras for $\mathcal{V}$ and continuous relations

- If R is a continuous relation on $X \in$ ***KHaus***, then $\rho : X \to \mathcal{V}(X)$ given by $\rho(x) := R[x]$ is a continuous function.

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Theorem (Folklore)

KHF *is isomorphic to* ***Coalg***$(\mathcal{V})$.

Algebraic/coalgebraic point of view

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This yields an alternate proof of Jónsson-Tarski duality.

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$$r = \|\alpha(x)\| \leq \|x\| < r.$$

The obtained contradiction proves that $F(X)$ does not exist.

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- Let ***WSet*** be the category whose objects are weighted sets and whose morphisms are functions $f : (X_1, w_1) \to (X_2, w_2)$ satisfying $w_2(f(x)) \leq w_1(x)$ for each $x \in X$.

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If $A \in \boldsymbol{ba\ell}$, then $(A, \|\cdot\|) \in \boldsymbol{WSet}$ and any morphism in $\boldsymbol{ba\ell}$ is a morphism in $\boldsymbol{WSet}$. Therefore, there is a forgetful functor $U : \boldsymbol{ba\ell} \to \boldsymbol{WSet}$.

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$F(X, w)$ is obtained by quotienting the free ℓ-algebra over X.

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- *$\mathbf{Alg}(\mathcal{H})$ is isomorphic to $\boldsymbol{mba\ell}$.*
- *There is a dual adjunction between $\mathbf{Alg}(\mathcal{H})$ over $\boldsymbol{ba\ell}$ and $\mathbf{Coalg}(\mathcal{V})$ over $\mathbf{KHaus}$.*

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This yields an alternate proof of the dual adjunction between $\boldsymbol{mba\ell}$ and $\boldsymbol{KHF}$, and of the dual equivalence between $\boldsymbol{muba\ell}$ and $\boldsymbol{KHF}$.

Adjunction and duality via algebras/coalgebras

$$\begin{array}{ccc} \boldsymbol{mba\ell} \cong \boldsymbol{Alg}(\mathcal{H}) & \rightleftarrows & \boldsymbol{Coalg}(\mathcal{V}) \cong \boldsymbol{KHF} \\ \uparrow & \swarrow\!\!\nearrow & \\ \boldsymbol{muba\ell} \cong \boldsymbol{Alg^{u}}(\mathcal{H}) & & \end{array}$$

Table of Contents

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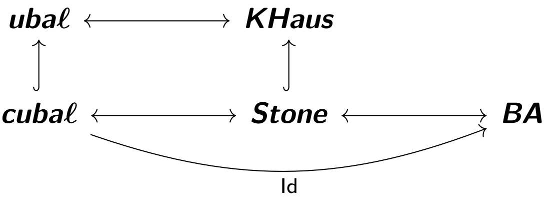

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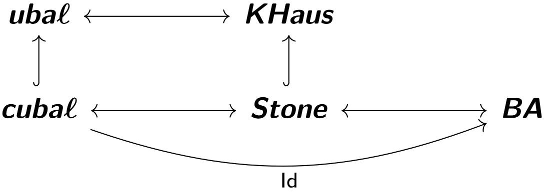

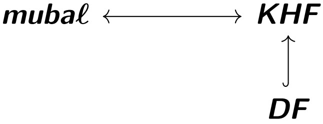

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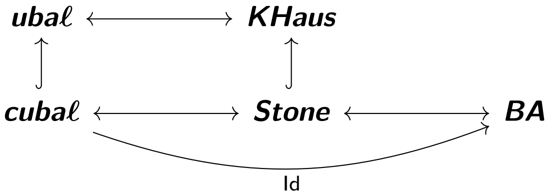

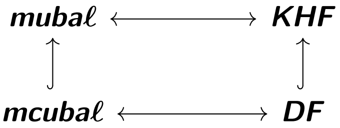

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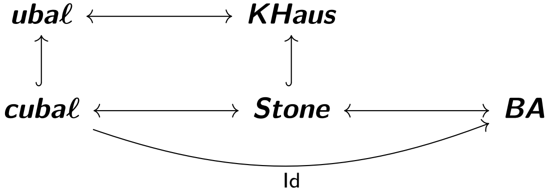

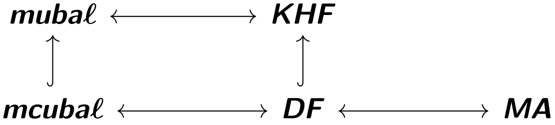

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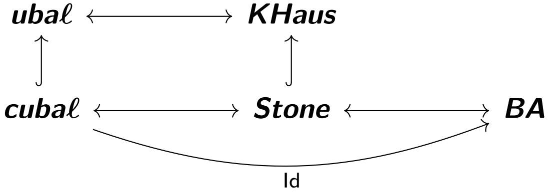

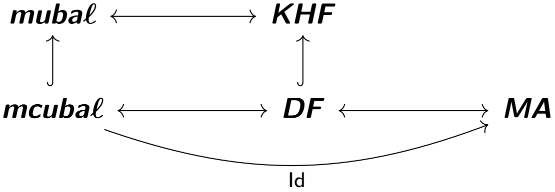

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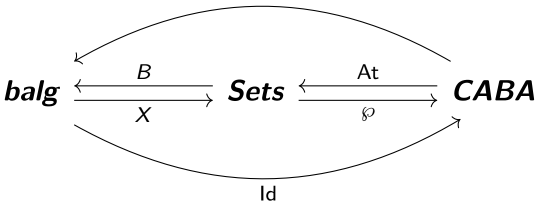

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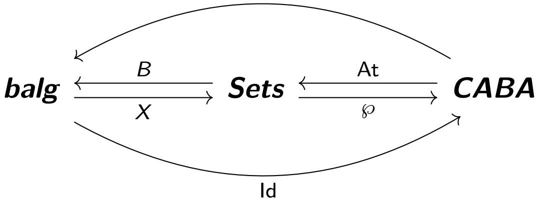

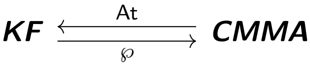

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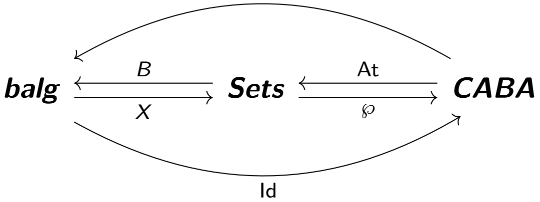

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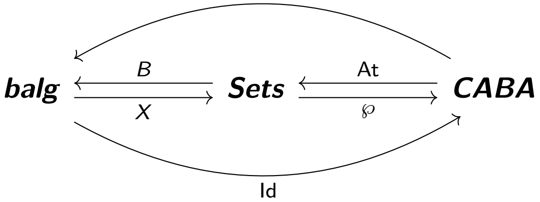

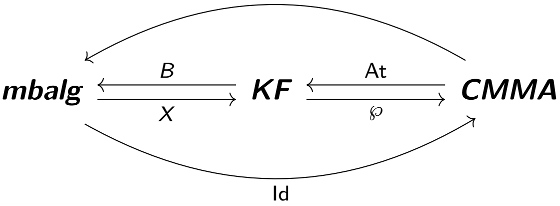

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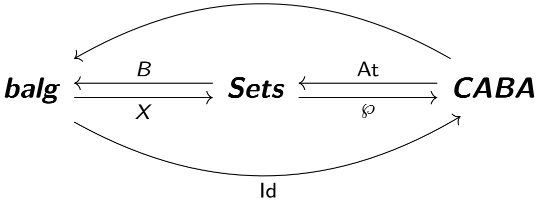

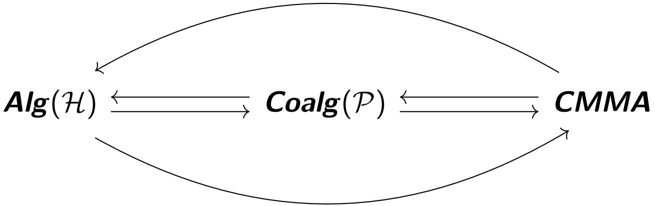

Correspondence theory

Let $(A, \Box) \in \boldsymbol{mba\ell}$. It turns out that $(A, \Box)$ satisfies the axiom on the right iff $R_\Box$ on Y_A satisfies the property on the left.

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- All the axioms considered above are preserved in the canonical extension.

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Dualities for compact Hausdorff frames extending these two dualities were investigated by G. Bezhanishvili, N. Bezhanishvili, and Harding (2015). They are obtained by endowing compact regular frames and de Vries algebras with modal operators. An interesting direction of research is to investigate the connections between these dualities for ***KHaus*** and ***KHF*** with Gelfand duality and its modal extension.

Thanks for your attention!

Stone duality and Gelfand duality

Definition

- A uniformly complete bounded archimedean ℓ-algebra A is called clean if each element of A can be written as a sum of an idempotent and a unit.
- The full subcategory of $\boldsymbol{uba\ell}$ given by its clean objects is denoted by $\boldsymbol{cuba\ell}$.
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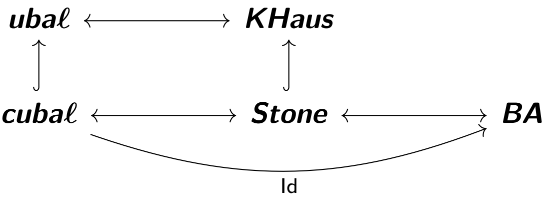

Esakia-Goldblatt duality and Gelfand duality

Definition

Let $\boldsymbol{mcuba\ell}$ the full subcategory of clean objects of $\boldsymbol{muba\ell}$.

Theorem

- *$\boldsymbol{mcuba\ell}$ is dually equivalent to the category of descriptive frames* $\boldsymbol{DF}$.

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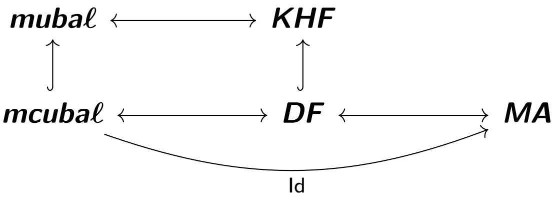

Basic algebras

Definition

- $A \in \boldsymbol{ba\ell}$ is Dedekind complete if each subset bounded above has a least upper bound, and hence each subset bounded below has a greatest lower bound.
- For $A \in \boldsymbol{ba\ell}$ let $\mathrm{Id}(A)$ be the boolean algebra of idempotents of A.
- We call $A \in \boldsymbol{ba\ell}$ a basic algebra if A is Dedekind complete and $\mathrm{Id}(A)$ is atomic.
- Let ***balg*** be the category of basic algebras and normal homomorphisms, i.e. the morphisms in $\boldsymbol{ba\ell}$ preserving all the existing joins and meets.

Proposition

Every basic algebra is uniformly complete.

balg and ***Sets***

Definition

Let $A \in$ ***balg*** and $X \in$ ***Sets***.

- let X_A be the set of co-atoms of $\mathrm{Id}(A)$. This yields a contravariant functor ***balg*** $\to$ ***Sets***.
- the set $B(X)$ of all bounded functions on X form naturally a basic algebra. This yields a contravariant functor ***Sets*** $\to$ ***balg***.

The following theorem can be thought of as an analogue of Tarski duality between the category of complete and atomic boolean algebras and ***Sets***.

Theorem

balg *is dually equivalent to* ***Sets***.

Modal basic algebras

Definition

- $(A, \Box) \in \boldsymbol{mba\ell}$ is a modal basic algebra if $A \in$ ***balg*** and $\Box$ preserves all the existing meets.
- Let ***mbalg*** be the category of modal basic algebras and normal homomorphisms preserving the modal operator.
- A Kripke frame (X, R) is a set X together with a binary relation R on X.
- We denote the category of Kripke frames and p-morphisms by ***KF***

The following theorem can be thought of as an analogue of Thomason duality between the category of completely multiplicative modal algebras and ***Sets***.

Theorem

mbalg *is dually equivalent to* ***KF***.

Duality between *mbalg* and *KF*

The duality can be obtained in two ways:

- by adapting the proof for ***mba***ℓ, or
- by using algebraic/coalgebraic methods.

Definition

- For $(X, R) \in$ ***KF*** we define $\Box_R$ on $B(X)$ as before. This defines a contravariant functor ***KF*** $\to$ ***mbalg***.
- For $A \in$ ***mbalg***, we define $R_\Box$ on X_A by $xR_\Box y$ iff $\Box y \leq x$. This defines a contravariant functor ***mbalg*** $\to$ ***KF***.

These two functors yield a dual equivalence between ***mbalg*** and ***KF***.

Duality between *mbalg* and *KF* using algebras and coalgebras

- ***KF*** is isomorphic to the category of coalgebras for the powerset endofunctor $\mathcal{P}$ on ***Sets***.

Theorem

- *There is an endofunctor $\mathcal{H}$ on **balg** so that **mbalg** is isomorphic to the category of algebras for $\mathcal{H}$.*
- ***Coalg**$(\mathcal{P})$ is dually equivalent to **Alg**$(\mathcal{H})$.*