Dualities for MV-algebras

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Łukasiewicz logic

Fuzzy logic

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Łukasiewicz logic, introduced by Jan Łukasiewicz in 1930, allows formulas to take truth values in the real unit interval $[0, 1]$.

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Łukasiewicz logic

The language of Łukasiewicz logic has the connectives

- Falsity 0 $\quad 0 := \neg(p \to p)$
- Truth 1 $\quad 1 := p \to p$
- Weak conjunction $\wedge \quad \varphi \wedge \psi := \neg((\varphi \to \psi) \to \neg\varphi)$
- Weak disjunction $\vee \quad \varphi \vee \psi := (\varphi \to \psi) \to \psi$
- Negation $\neg$
- Strong conjunction $\odot \quad \varphi \odot \psi := \neg(\varphi \to \neg\psi)$
- Strong disjunction $\oplus \quad \varphi \oplus \psi := \neg\varphi \to \psi$
- Implication $\to$

Łukasiewicz introduced the logic as a logical calculus using $\to$ and $\neg$ as primitive connectives.

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- Falsity 0
- Truth 1 $\quad 1 := \neg 0$
- Weak conjunction $\wedge$ $\quad \varphi \wedge \psi := \neg(\neg\varphi \oplus \neg(\neg\varphi \oplus \psi))$
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Another option is to use 0, $\neg$, and $\oplus$ as primitive connectives.

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Theorem (Completeness)

φ is a theorem of Łukasiewicz logic iff $v(\varphi) = 1$ for each valuation v.

MV-algebras

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Definition (Chang 1958)

An *MV-algebra* is a structure $(A, \oplus, \neg, 0)$ satisfying

1. $x \oplus (y \oplus z) = (x \oplus y) \oplus z$
2. $x \oplus y = y \oplus z$
3. $x \oplus 0 = x$
4. $\neg\neg x = x$
5. $x \oplus \neg 0 = \neg 0$
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Since MV-algebras are defined by equations, the class of MV-algebras form a variety i.e. it is closed under products, subalgebras, and homomorphic images.

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Theorem (Algebraic semantics)

Let φ be a formula and t the corresponding term. φ is a theorem of Łukasiewicz logic iff the equation $t = 1$ is true in every MV-algebra.

The MV-algebra $[0, 1]$

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As a consequence

- $x \wedge y = \min\{x, y\}$
- $x \vee y = \max\{x, y\}$
- $x \odot y = \max\{0, x + y - 1\}$
- $x \to y = \min\{1, 1 - x + y\}$

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We can think of this fact as an algebraic way to state the completeness of Łukasiewicz logic with respect to the $[0, 1]$-valued semantics.

Examples of MV-algebras

Boolean algebras

They are exactly the MV-algebras in which $\wedge = \odot$ (or $\vee = \oplus$). It follows that every theorem of Łukasiewicz logic (that doesn't contain $\odot$ and $\oplus$) is a classical tautology.

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$C(X)$

The set of all continuous functions from a topological space X into $[0,1]$ with pointwise operations.

Examples of MV-algebras: $\mathrm{PWL}_{\mathbb{Z}}(X)$

A continuous function $f : [0,1]^\kappa \to [0,1]$ is piecewise linear if there exist $g_1, \ldots, g_n$ polynomials of degree one in the variables $(x_\alpha)_{\alpha<\kappa}$ such that for each $x \in [0,1]^\kappa$ we have $f(x) = g_i(x)$ for some i.

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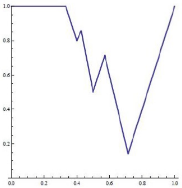
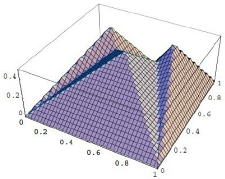
$\mathrm{PWL}_{\mathbb{Z}}([0,1]^\kappa)$ is the set of all piecewise linear functions such that $g_1, \ldots, g_n$ have integer coefficients. These functions are also known as $\mathbb{Z}$-maps or MacNaughton functions.





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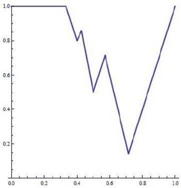
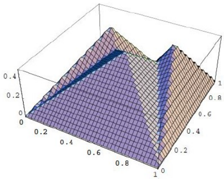
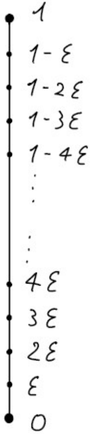




If $X \subseteq [0,1]^\kappa$, we denote by $\mathrm{PWL}_{\mathbb{Z}}(X)$ the set of maps in $\mathrm{PWL}_{\mathbb{Z}}([0,1]^\kappa)$ restricted to X. It is an MV-algebra with pointwise operations.



ε is an infinitesimal element.

Indeed, $n\varepsilon \leq 1 - \varepsilon$ for every n.

Definition

An abelian ℓ-group G is an abelian group equipped with a lattice order such that $a \leq b$ implies $a + c \leq b + c$ for all $a, b, c \in G$.

$u \in G$ is a strong order-unit if for each $x \in G$ there is $n \in \mathbb{N}$ such that $x \leq nu$.

MV-algebras and ℓ-groups

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In fact, every MV-algebra arises in this way.

Theorem (Mundici 1986)

The category of abelian ℓ-groups with strong order-unit is equivalent to the category of MV-algebras.

Ideals

Definition

A nonempty subset I of an MV-algebra A is an ideal if

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Proper ideals that are maximal wrt the inclusion are called maximal ideals. We denote by $\mathrm{Max}(A)$ the set of maximal ideals of A.

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Proposition

Let I be an ideal of an MV-algebra A. Then

- A/I is simple iff I is maximal.
- A/I is semisimple iff I is intersection of maximal ideals.

Duality for semisimple MV-algebras

Simple MV-algebras and $[0,1]$

Proposition

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What about semisimple MV-algebras?

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$[0,1]^X$ and all its subalgebras are semisimple.

Representation of semisimple MV-algebras

Theorem

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Each $M \in \mathrm{Max}(A)$ is such that A/M embeds into $[0,1]$. Thus, if $a \in A$, we can define a map $\mathrm{Max}(A) \to [0,1]$ by associating to M the image of a/M under the embedding $A/M \hookrightarrow [0,1]$.

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Since the intersection of all maximal ideals of A is $\{0\}$, this defines an embedding $A \hookrightarrow [0,1]^{\mathrm{Max}(A)}$.

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Suppose that $(g_\alpha)_{\alpha<\kappa}$ are generators of A. If $M \in \mathrm{Max}(A)$, we send M to (r_α) where r_α is the image of g_α/M under the embedding $A/M \hookrightarrow [0,1]$.

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Theorem

If A is semisimple, then A is isomorphic to $\mathrm{PWL}_{\mathbb{Z}}(\mathrm{Max}(A))$.

Applying a general duality approach due to Caramello, Marra, and Spada it is possible to obtain a duality for semisimple MV-algebras.

Theorem (Marra-Spada 2012)

The category of semisimple MV-algebras is dually equivalent to the category of closed subsets of Tychnoff cubes.

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Vice versa, to a closed subset C of $[0,1]^{\kappa}$ we associate MV-algebra $\mathrm{PWL}_{\mathbb{Z}}(C)$.

Examples

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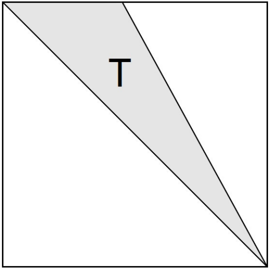

$\mathrm{PWL}_{\mathbb{Z}}(T)$

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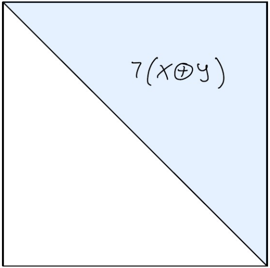

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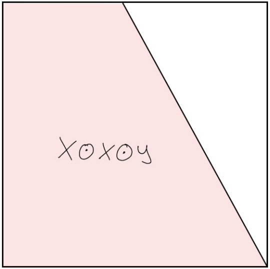

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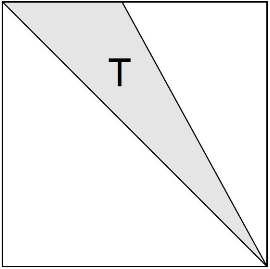

$$\mathrm{PWL}_{\mathbb{Z}}(T) \cong \frac{\mathcal{F}(x, y)}{\langle \neg(x \oplus y), x \odot x \odot y \rangle}$$

Extending the duality to all MV-algebras

Prime ideals

Our goal is to extend the duality for semisimple MV-algebras to all MV-algebras by working with prime ideals instead of maximal ideals.

Definition

A proper ideal I of an MV-algebra A is prime if $x \wedge y \in I$ implies $x \in I$ or $y \in I$. We denote by $\mathrm{Spec}(A)$ the set of all prime ideals of A.

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Proposition

Let A be an MV-algebra and I a proper ideal of A. We have that

- I is prime iff A/I is linearly ordered.
- The intersection of all prime ideals of A is $\{0\}$.
- A embeds into a product of linearly ordered MV-algebras.

Di Nola Theorem

We need an MV-algebra in which we can embed the linearly ordered MV-algebras.

Theorem (Di Nola)

Let γ be an infinite cardinal. Then there exists an ultrapower $\mathcal{U}$ of the MV-algebra $[0,1]$ such that every linearly ordered MV-algebra A with $|A| \leq \gamma$ embeds into $\mathcal{U}$.

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$\mathcal{U}$ is a linearly ordered MV-algebra containing $[0,1]$ and lots of infinitesimals. Any $f \in \mathrm{PWL}_{\mathbb{Z}}([0,1]^{\kappa})$ can be extended to a function ${}^{*}f : \mathcal{U}^{\kappa} \to \mathcal{U}$.

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$\mathcal{U}^\kappa$ can be endowed with the Zariski topology which is given by a basis of closed consisting of the sets $\{x \in \mathcal{U}^\kappa \mid {}^*f(x) = 0\}$ where f ranges in $\mathrm{PWL}_{\mathbb{Z}}([0,1]^\kappa)$. This topology is compact but not even T_0.

Coordinatize Spec(A)

What happens if we try to coordinatize Spec(A) like we did with Max(A)? For the embedding $A/P \hookrightarrow \mathcal{U}$ to exist we need $|A/P| \leq \gamma$ and the embedding is not necessarily unique.

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Suppose that $(g_\alpha)_{\alpha<\kappa}$ are generators of A with $\kappa \leq \gamma$. If $P \in \text{Spec}(A)$, we send P to the set of all the $(r_\alpha) \in \mathcal{U}^\kappa$ for which there exists an embedding $A/P \hookrightarrow \mathcal{U}$ that maps g_α/M to r_α for all α.

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So we can map Spec(A) to the union of all these subsets of $\mathcal{U}^\kappa$.
This union turns out to be a closed subset of $\mathcal{U}^\kappa$.
In particular when A is the free MV-algebra on κ generators, Spec(A) corresponds to the whole $\mathcal{U}^\kappa$.

Extending the duality to all MV-algebras

Applying the general duality approach of Caramello, Marra, and Spada it is possible to obtain also a duality for all MV-algebras.

Theorem (Carai-Lapenta-Spada)

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Vice versa, to a closed subset C of $\mathcal{U}^\kappa$ we associate the MV-algebra ${}^*\mathrm{PWL}_{\mathbb{Z}}(C)$ given by the restrictions to C of all the ${}^*f$ with $f \in \mathrm{PWL}_{\mathbb{Z}}([0,1]^\kappa)$.

Coordinatization of Spec(A)

We would like to coordinatize Spec(A) (i.e. embed it into $\mathcal{U}^\kappa$) so that $A \cong {}^*\mathrm{PWL}_{\mathbb{Z}}(\mathrm{Spec}(A))$.

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Theorem

If $x \in \mathcal{U}^n$, then $x = x_0 + \alpha_1 v_1 + \cdots + \alpha_t v_t$ where $\alpha_1, \ldots, \alpha_t \in \mathcal{U}$ are positive infinitesimals such that α_{i+1}/α_i is infinitesimal, $x_0 \in [0,1]^n$, and $v_1, \ldots, v_t$ are orthonormal vectors in $\mathbb{R}^n$.

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Let x be any of the points of the "infinitesimal simplex" inside $\mathcal{U}^n$ associated to P. Suppose $x = x_0 + \alpha_1 v_1 + \cdots + \alpha_t v_t$.

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Apply a sort of Gram-Schmidt process to $(x_0, v_1, \ldots, v_t)$ to obtain $(x_0, w_1, \ldots, w_s)$.

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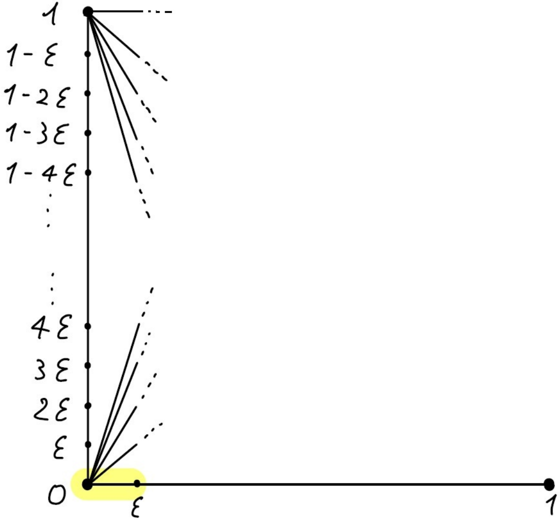
Fix any infinitesimal ε and then associate to P the point $x_0 + \varepsilon w_1 + \cdots + \varepsilon^s w_s \in \mathcal{U}^n$.

Example: Chang



Other dualities

Riesz MV-algebras

Definition

A Riesz MV-algebra is a structure $(R, \cdot)$ where R is an MV-algebra and $\cdot : [0,1] \times R \to R$ is such that

1. If $x \odot y = 0$, then $(rx) \odot (ry) = 0$ and $r(x \oplus y) = rx \oplus ry$.
2. If $r \odot q = 0$, then $(rx) \odot (qx) = 0$ and $(r \oplus q)x = rx \oplus ry$.
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We proved an analogous duality result in this setting. The main differences are that the piecewise linear functions can have non-integer coefficients and hence the Zariski topology on $\mathcal{U}^\kappa$ is finer.

Abelian ℓ-groups and Riesz spaces

We also proved an analogous duality for abelian l-groups and Riesz spaces (real vector lattices). This result generalizes the Baker-Beynon duality.

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Here are the main differences from the MV-algebras and Riesz MV-algebras case:

- Instead of working with an ultrapower of $[0, 1]$, we have an ultrapower of $\mathbb{R}$.
- The linear pieces of piecewise linear functions are homogeneous.
- Infinitesimal simplexes are replaced by infinitesimal cones.

THANK YOU!