Coalgebras for the powerset functor and Thomason duality

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Dualities for modal algebras

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- Jónsson-Tarski duality which is a generalization of Stone duality.
- Thomason duality which is a generalization of Tarski duality.

Stone and Tarski dualities

Definition

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Definition

- If B is a complete boolean algebra, we say that a unary operator $\Box$ on B is completely multiplicative if it preserves arbitrary meets.
- **CAMA** is the category with
  - *objects:* complete and atomic boolean algebras equipped with a completely multiplicative operator $\Box$
  - *morphisms:* complete boolean homomorphisms preserving $\Box$.

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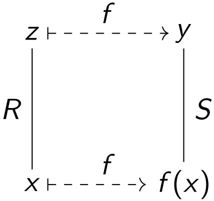
- A map between two Kripke frames (X, R) and (Y, S) is called a p-morphism if for all $x \in X$ and $y \in Y$, we have

$$f(x)Sy \;\Leftrightarrow\; \exists z \in X \; (xRz \text{ and } f(z) = y)$$



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- **KFr** is the category of Kripke frames and p-morphisms.

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The category of descriptive frames and continuous p-morphisms is denoted by **DFr**.

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The contravariant functors yielding the dual equivalence are:

- $\mathfrak{uf} : \mathbf{MA} \to \mathbf{DFr}$ associating with each $(B, \Box) \in \mathbf{MA}$ the Stone space of its ultrafilters with the relation $R_\Box$ defined by

$$xR_\Box y \quad \text{iff} \quad \forall a \in B \ (\Box a \in x \Rightarrow a \in y)$$

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- $\mathrm{Clop} : \mathbf{DFr} \to \mathbf{MA}$ associating with each $(X, R) \in \mathbf{DFr}$ the set of clopens of X equipped with the operator $\Box_R$ defined for every clopen subset A of X

$$\Box_R(A) = \{x \in X \mid R[x] \subseteq A\}$$

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- $\wp$: **KFr** $\to$ **CAMA** associating with each $(X, R) \in$ **KFr** the powerset $\wp(X)$ of X equipped with the operator $\Box_R$ defined for every $A \subseteq X$ by

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$$\Diamond_U = \{F \in \mathcal{V}(X) \mid F \cap U \neq \emptyset\}$$
$$\Box_K = \{F \in \mathcal{V}(X) \mid F \cap K = \emptyset\}$$

where U ranges over all open subsets of X and K over all the closed subsets.

Descriptive frames as coalgebras

Theorem (Michael)

*If $X \in$ **Stone**, then $\mathcal{V}(X) \in$ **Stone**. Moreover, $\mathcal{V}$ is an endofunctor on **Stone**.*

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- Given a continuous function $\rho : X \to \mathcal{V}(X)$, we associate the continuous binary relation R_ρ defined by $xR_\rho y$ iff $y \in \rho(x)$.

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- Given a continuous function $\rho : X \to \mathcal{V}(X)$, we associate the continuous binary relation R_ρ defined by $xR_\rho y$ iff $y \in \rho(x)$.

Thus, descriptive frames can be thought of as coalgebras for the Vietoris endofunctor $\mathcal{V}$ on **Stone**.

Coalgebras

Definition

- A coalgebra for an endofunctor $\mathcal{T} : \mathbf{C} \to \mathbf{C}$ is a pair (C, g) where C is an object of $\mathbf{C}$ and $g : C \to \mathcal{T}(C)$ is a $\mathbf{C}$-morphism.

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- A morphism between (C_1, g_1) and (C_2, g_2) is a $\mathbf{C}$-morphism $\alpha : C_1 \to C_2$ such that the following square is commutative.

$$
\begin{array}{ccc}
C_1 & \xrightarrow{\quad\alpha\quad} & C_2 \\
{\scriptstyle g_1}\big\downarrow & & \big\downarrow{\scriptstyle g_2} \\
\mathcal{T}(C_1) & \xrightarrow[\mathcal{T}(\alpha)]{} & \mathcal{T}(C_2)
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- Morphisms between algebras and the category $\mathbf{Alg}(\mathcal{T})$ of algebras for $\mathcal{T}$ are defined analogously.

Jónsson-Tarski duality via algebras/coalgebras

Proposition

DFr *is isomorphic to the category* **Coalg**$(\mathcal{V})$ *of coalgebras for the Vietoris endofunctor on* **Stone**.

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This yields an alternative proof of Jónsson-Tarski duality.

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Thus, the elements of the form $\Box_a$ generate $\mathcal{K}(B)$ and we have $\Box_1 = 1$ and $\Box_{a \wedge b} = \Box_a \wedge \Box_b$.

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Definition

Let $\mathcal{P} : \mathbf{Set} \to \mathbf{Set}$ be the covariant powerset functor that maps

- each set X to its powerset $\mathcal{P}(X)$, and
- each function $f : X \to Y$ to $\mathcal{P}(f) : \mathcal{P}(X) \to \mathcal{P}(Y)$ sending $S \subseteq X$ to its image $f[S] \subseteq Y$.

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A correspondence analogous to the one for descriptive frames yields

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This yields an alternative proof of Thomason duality.

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Another reason why **CABA** has free objects is because it is isomorphic to the Eilenberg-Moore category of the double powerset monad.

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$\mathbf{Alg}(\mathcal{H})$ and $\mathbf{Coalg}(\mathcal{P})$

The following diagram commutes up to natural isomorphism.

$$\begin{array}{ccc} \mathbf{CABA} & \xleftrightarrow{\text{Tarski}} & \mathbf{Set} \\ \downarrow \mathcal{H} & & \downarrow \mathcal{P} \\ \mathbf{CABA} & \xleftrightarrow{\text{Tarski}} & \mathbf{Set} \end{array}$$

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We obtain as a consequence that

Theorem

Tarski duality lifts to a duality between $\mathbf{Alg}(\mathcal{H})$ *and* $\mathbf{Coalg}(\mathcal{P})$.

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- **CAMA** is isomorphic to **Alg**$(\mathcal{H})$.

$$\mathbf{CAMA} \overset{\cong}{\longleftrightarrow} \mathbf{Alg}(\mathcal{H}) \qquad \mathbf{Coalg}(\mathcal{P}) \overset{\cong}{\longleftrightarrow} \mathbf{KFr}$$

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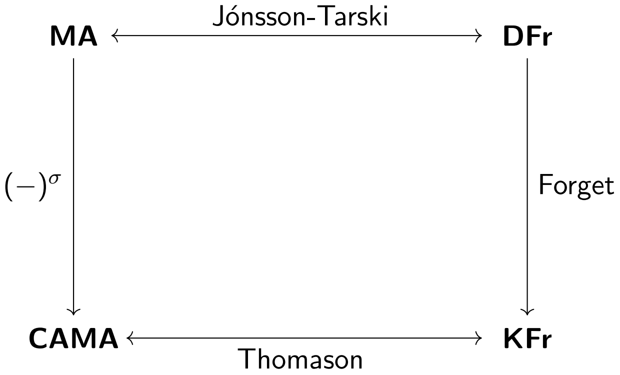
$$\mathbf{CAMA} \xleftrightarrow{\ \cong\ } \mathbf{Alg}(\mathcal{H}) \xleftrightarrow{\ \cong^{\mathsf{op}}\ } \mathbf{Coalg}(\mathcal{P}) \xleftrightarrow{\ \cong\ } \mathbf{KFr}$$

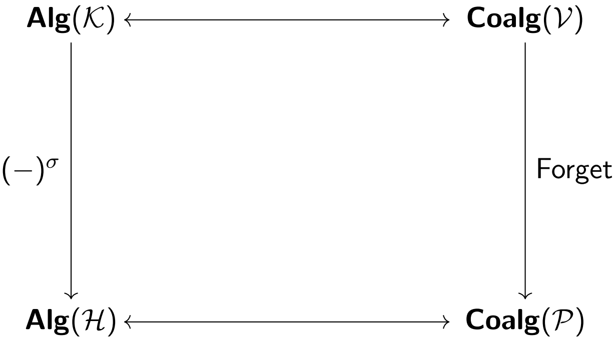
$$\mathfrak{at} : \mathbf{CAMA} \to \mathbf{KFr}, \qquad \wp : \mathbf{KFr} \to \mathbf{CAMA}$$

By composition we obtain the functors of Thomason duality.

Connecting the dualities



Connecting the dualities



THANK YOU!