Dualities for abelian ℓ-groups and vector lattices beyond archimedeanity

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joint work with S. Lapenta and L. Spada

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Abelian ℓ-groups and vector lattices

Definition

- An abelian ℓ-group is an abelian group A equipped with a lattice order such that $a \leq b$ implies $a + c \leq b + c$ for every $a, b, c \in A$.
- A vector lattice is an abelian ℓ-group V equipped with a structure of $\mathbb{R}$-vector space such that $0 \leq r$ and $0 \leq v$ imply $rv \geq 0$ for each $r \in \mathbb{R}$ and $v \in V$.

Abelian ℓ-groups and vector lattices form varieties.

ℓ-ideals

Definition

- An ℓ-ideal in an abelian ℓ-group is a subgroup I that is convex, i.e. $|a| \leq |b|$ and $b \in I$ imply $a \in I$.
- An ℓ-ideal in a vector lattice is a vector subspace that is convex.

Definition

An abelian ℓ-group/vector lattice is semisimple if the intersection of all its maximal ℓ-ideals is $\{0\}$.

It is archimedean if $na \leq b$ for every $n \in \mathbb{N}$ implies $a \leq 0$.

Semisimple $\Rightarrow$ archimedean

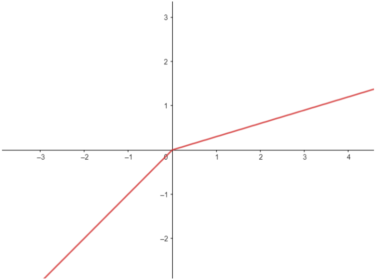
Archimedean $\Rightarrow$ semisimple (if finitely generated)

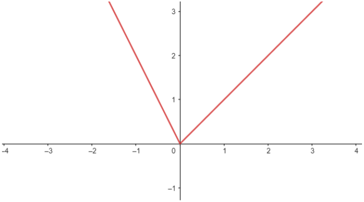
Baker-Beynon duality

Piecewise linear functions

Definition

A continuous function $f : \mathbb{R}^\kappa \to \mathbb{R}$ is piecewise linear if there exist $g_1, \ldots, g_n$ linear homogeneous polynomials in the variables $(x_\alpha)_{\alpha<\kappa}$ such that for each $x \in \mathbb{R}^\kappa$ we have $f(x) = g_i(x)$ for some $i = 1, \ldots, n$.





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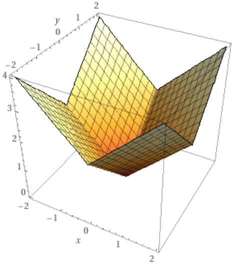
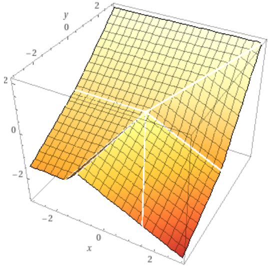




Piecewise linear functions

- The set $\mathrm{PWL}_{\mathbb{R}}(\mathbb{R}^\kappa)$ of piecewise linear functions on $\mathbb{R}^\kappa$ is a vector lattice with pointwise operations.
- The set $\mathrm{PWL}_{\mathbb{Z}}(\mathbb{R}^\kappa)$ of piecewise linear functions on $\mathbb{R}^\kappa$ such that $g_1, \ldots, g_n$ have integer coefficients is an abelian ℓ-group with pointwise operations.

Theorem

- $\mathrm{PWL}_{\mathbb{R}}(\mathbb{R}^\kappa)$ *is iso to the free vector lattice on* κ *generators.*
- $\mathrm{PWL}_{\mathbb{Z}}(\mathbb{R}^\kappa)$ *is iso to the free abelian* ℓ*-group on* κ *generators.*

Piecewise linear functions

If $X \subseteq \mathbb{R}^\kappa$, we denote by $\mathrm{PWL}_{\mathbb{R}}(X)$ and $\mathrm{PWL}_{\mathbb{Z}}(X)$ the sets of piecewise linear maps restricted to X.

Definition

A subset of $\mathbb{R}^\kappa$ is a cone if it is closed under multiplication by nonnegative scalars.

Theorem (Baker 1968)

- *Every κ-generated semisimple vector lattice is isomorphic to $\mathrm{PWL}_{\mathbb{R}}(C)$ where C is a cone that is closed in $\mathbb{R}^\kappa$.*
- *Every κ-generated semisimple abelian ℓ-group is isomorphic to $\mathrm{PWL}_{\mathbb{Z}}(C)$ where C is a cone that is closed in $\mathbb{R}^\kappa$.*

Baker-Beynon duality

$\mathrm{PWL}_{\mathbb{R}}$ and $\mathrm{PWL}_{\mathbb{Z}}$ can be extended to contravariant functors. They yield the Baker-Beynon duality.

Theorem (Beynon 1974)

- *The category of semisimple vector lattices is dually equivalent to the category of closed cones in $\mathbb{R}^\kappa$ and piecewise linear maps with real coefficients.*
- *The category of semisimple abelian ℓ-groups is dually equivalent to the category of closed cones in $\mathbb{R}^\kappa$ and piecewise linear maps with integer coefficients.*

Beyond Baker-Beynon duality

Key ingredient for Baker-Beynon

Any semisimple abelian ℓ-group/vector lattice is a subdirect product of subalgebras of $\mathbb{R}$.

We need an analogous fact without the semisimplicity assumption. However, we are forced to put a bound on the cardinality of the set of generators.

Theorem

Let γ be a cardinal. There exists an ultrapower $\mathcal{U}$ of $\mathbb{R}$ such that for any $\kappa \leq \gamma$ any κ-generated abelian ℓ-group/vector lattice is a subdirect product of subalgebras of $\mathcal{U}$.

$\mathcal{U}$ and $\mathbb{R}$ are elementary equivalent but $\mathcal{U}$ contains infinitesimals and infinite elements. Moreover, $\mathcal{U}$ contains a copy of $\mathbb{R}$.

Extending piecewise linear maps and Zariski topologies on $\mathcal{U}^\kappa$

We can extend every piecewise linear $f : \mathbb{R}^\kappa \to \mathbb{R}$ to ${}^*f : \mathcal{U}^\kappa \to \mathcal{U}$ which is called the enlargement of f.

We define:
${}^*\mathrm{PWL}_{\mathbb{R}}(\mathcal{U}^\kappa) = \{{}^*f \mid f \in \mathrm{PWL}_{\mathbb{R}}(\mathbb{R}^\kappa)\}$,
${}^*\mathrm{PWL}_{\mathbb{Z}}(\mathcal{U}^\kappa) = \{{}^*f \mid f \in \mathrm{PWL}_{\mathbb{Z}}(\mathbb{R}^\kappa)\}$.

If $X \subseteq \mathcal{U}^\kappa$, we can consider the restriction of ${}^*f$ to X. We denote the corresponding sets by ${}^*\mathrm{PWL}_{\mathbb{R}}(X)$ and ${}^*\mathrm{PWL}_{\mathbb{Z}}(X)$.

The closed cones in $\mathbb{R}^\kappa$ are the closed subsets of the topology generated by the zerosets of the maps in $\mathrm{PWL}_{\mathbb{R}}(\mathbb{R}^\kappa)$, or equivalently $\mathrm{PWL}_{\mathbb{Z}}(\mathbb{R}^\kappa)$.

We consider the topologies on $\mathcal{U}^\kappa$ generated by the zerosets of the maps in ${}^*\mathrm{PWL}_{\mathbb{R}}(\mathcal{U}^\kappa)$ and ${}^*\mathrm{PWL}_{\mathbb{Z}}(\mathcal{U}^\kappa)$. These two topologies do not coincide. We call them Zariski topologies.

By applying the general affine duality approach of Caramello, Marra, and Spada (2021) we obtain a duality.

Theorem (C., Lapenta, and Spada)

Let γ be a cardinal. There exists an ultrapower $\mathcal{U}$ of $\mathbb{R}$ such that:

- *The category of κ-generated vector lattices for some $\kappa \leq \gamma$ is dually equivalent to the category of Zariski closed subsets of $\mathcal{U}^\kappa$ for some $\kappa \leq \gamma$.*
- *The category of κ-generated abelian ℓ-groups for some $\kappa \leq \gamma$ is dually equivalent to the category of Zariski closed subsets of $\mathcal{U}^\kappa$ for some $\kappa \leq \gamma$.*

Thus, every vector lattice (resp. abelian ℓ-group) is isomorphic to $^*\mathrm{PWL}_{\mathbb{R}}(C)$ (resp. $^*\mathrm{PWL}_{\mathbb{Z}}(C)$) for some Zariski closed subset $C \subseteq \mathcal{U}^\kappa$.

Correspondence between ℓ-ideals and closed subsets

$\mathscr{F}_{\kappa}$	$\mathbb{R}^{\kappa}$	$\mathcal{U}^{\kappa}$
maximal ℓ-ideals	half-lines from the origin	closure of standard points (except the origin) $=$ half-lines from the origin through a standard point
intersections of maximal ℓ-ideals	closed cones	closure of standard subsets
prime ℓ-ideals		irreducible closed subsets $=$ closure of points
ℓ-ideals		closed subsets

Embedding the spectrum

The maximal spectrum MaxSpec(A) of a semisimple vector lattice/abelian ℓ-group A can be embedded into its dual closed cone in $\mathbb{R}^\kappa$.

If we identify MaxSpec(A) with its image, $A \cong \mathrm{PWL}_{\mathbb{R}}(\mathrm{MaxSpec}(A))$ or $A \cong \mathrm{PWL}_{\mathbb{Z}}(\mathrm{MaxSpec}(A))$.

Similarly, the spectrum Spec(A) of a vector lattice/abelian ℓ-group can be embedded into its dual Zariski closed subset of $\mathcal{U}^\kappa$

If we identify Spec(A) with its image, $A \cong {}^*\mathrm{PWL}_{\mathbb{R}}(\mathrm{Spec}(A))$ or $A \cong {}^*\mathrm{PWL}_{\mathbb{Z}}(\mathrm{Spec}(A))$.

Irreducible closed subsets of $\mathcal{U}^n$ (vector lattices)

Irreducible closed subsets of $\mathcal{U}^n$

Orthogonal decomposition theorem (Goze 1995)

If $x \in \mathcal{U}^n$, then x can be written in a unique way as $\alpha_1 v_1 + \cdots + \alpha_k v_k$ with $v_1, \ldots, v_k$ orthonormal vectors of $\mathbb{R}^n$ and $0 < \alpha_1, \ldots, \alpha_k \in \mathcal{U}$ such that α_{i+1}/α_i is infinitesimal.

Thus, we can associate to each $x \in \mathcal{U}^n$ the sequence $\mathbf{v} = (v_1, \ldots, v_k)$ of orthonormal vectors.
We call such sequences indices.

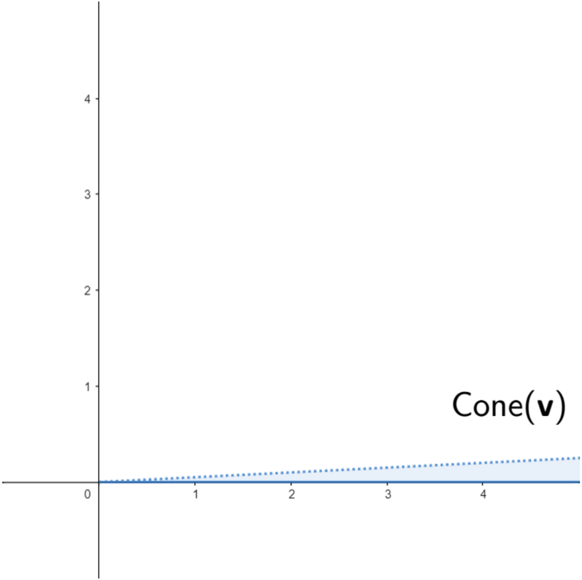
Definition

Let $\mathrm{Cone}(\mathbf{v})$ be the set of points of $\mathcal{U}^n$ whose index is a truncation of $\mathbf{v}$.

Theorem (C., Lapenta, Spada)

In the Zariski topology of $\mathcal{U}^n$ relative to vector lattices each irreducible closed of $\mathcal{U}^n$ is $\mathrm{Cone}(\mathbf{v})$ *for some index* $\mathbf{v}$.

$\mathbf{v} = ((1,0),(0,1)).$



Primes and indices

Let $\mathbf{v}$ be an index. A polyhedral cone C of $\mathbb{R}^n$ is a **$\mathbf{v}$-cone** if there are real numbers $r_2, \ldots, r_k > 0$ such that the edges of C are given by $v_1, \quad v_1 + r_2 v_2, \quad \ldots \quad v_1 + r_2 v_2 + \cdots + r_k v_k$.

By transfer principle (Łoś's Theorem) we obtain

Theorem (C., Lapenta, and Spada)

If $f \in \mathrm{PWL}_{\mathbb{R}}(\mathbb{R}^n)$, then ${}^*f$ vanishes on $\mathrm{Cone}(\mathbf{v})$ iff f vanishes on some $\mathbf{v}$-cone.

As a corollary, we obtain

Theorem (Panti 1999)

Each prime ℓ-ideal of the vector lattice $\mathscr{F}_n$ is of the form $\{f \in \mathrm{PWL}_{\mathbb{R}}(\mathbb{R}^n) \mid f \text{ vanishes on a } \mathbf{v}\text{-cone}\}$ for some index $\mathbf{v}$.

Dualities for MV-algebras and Riesz MV-algebras beyond archimedeanity

Theorem (C., Lapenta, and Spada)

Let γ be a cardinal. There exists an ultrapower $\mathcal{U}$ of $[0,1]$ such that:

- *The category of κ-generated MV-algebras for some $\kappa \leq \gamma$ is dually equivalent to the category of Zariski closed subsets of $\mathcal{U}^\kappa$ for some $\kappa \leq \gamma$.*
- *The category of κ-generated Riesz MV-algebras for some $\kappa \leq \gamma$ is dually equivalent to the category of Zariski closed subsets of $\mathcal{U}^\kappa$ for some $\kappa \leq \gamma$.*

The irreducible closed in $\mathcal{U}^n$ are "infinitesimal simplices".

This is an affine version of the dualities for abelian ℓ-groups and vector lattices.

THANK YOU!