From implicit to explicit definability in algebra and logic

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Implicit vs explicit definability

Suppose we are given:

- two first-order languages $\mathscr{L}$ and $\mathscr{L}^+$ with $\mathscr{L} \subseteq \mathscr{L}^+$;
- a theory T in $\mathscr{L}^+$;
- a formula $\varphi(x_1, \ldots, x_n)$ of $\mathscr{L}^+$.

Definition

T **implicitly defines** φ relative to $\mathscr{L}$ when: for every pair of models A and B of T such that $A{\restriction_{\mathscr{L}}} = B{\restriction_{\mathscr{L}}}$, for all $a_1, \ldots, a_n \in A$,

$$A \vDash \varphi(a_1, \ldots, a_n) \text{ iff } B \vDash \varphi(a_1, \ldots, a_n).$$

T implicitly defines φ relative to $\mathscr{L}$ when the validity of φ in models of T is completely determined by their $\mathscr{L}$-reducts.

Example

- Let $\mathscr{L} = \{+, \cdot, -, 0, 1\}$ and $\mathscr{L}^{+} = \mathscr{L} \cup \{\leq\}$;
- let $T = \mathrm{Th}(\mathbb{R}, +, \cdot, -, 0, 1) \cup \{\text{axioms of ordered fields}\}$;
- let $\varphi(x_1, x_2)$ be $x_1 \leq x_2$.

T implicitly defines φ relative to $\mathscr{L}$ because every real closed field admits a unique order turning it into an ordered field.

The order in models of T is completely determined by the rest of the structure.

Definition

T **explicitly defines** φ relative to $\mathscr{L}$ when there exists a formula $\psi(x_1, \ldots, x_n)$ of $\mathscr{L}$ such that φ is equivalent modulo T to ψ.

T explicitly defines φ relative to $\mathscr{L}$ when T proves that φ is equivalent to a formula in $\mathscr{L}$.

Example

T defines $x_1 \leq x_2$ explicitly relative to $\mathscr{L}$ because in ordered real closed fields it holds that

$$x_1 \leq x_2 \text{ iff } \exists y(x_2 - x_1 = y^2).$$

The order in models of T can be defined via a formula not containing $\leq$.

Theorem

Let $\mathscr{L}, \mathscr{L}^+, T, \varphi$ as before. In classical first-order logic, φ is implicitly definable by T relative to $\mathscr{L}$ iff it is explicitly definable.

In the context of propositional logics this property is known as the **projective Beth definability property**.

Corollary

Classical propositional logic has the projective Beth definability property.

When $\mathscr{L}^+ = \mathscr{L} \cup \{R\}$ and $\varphi(x_1, \dots, x_n)$ is $R(x_1, \dots, x_n)$, the theorem above is known as **Beth definability theorem** (1953).

When a propositional logic satisfies the Beth definability theorem, it is said to have the **(singleton) Beth definability property**.

A homomorphism $f\colon A \to B$ in a class of algebras K is called an **epimorphism** if it is right cancellable; i.e., for every $g, h\colon B \to C$ with $C \in \mathsf{K}$

$$gf = hf \text{ implies } g = h.$$

Definition

K has the **weak epimorphism surjectivity (weak ES) property** if every epimorphism between finitely generated members of K is surjective.

L. Carai, M. Kurtzhals, and T. Moraschini, *Epimorphisms between finitely generated algebras*. Indagationes Mathematicae (N.S.), 36(5):1336–1354, 2025.

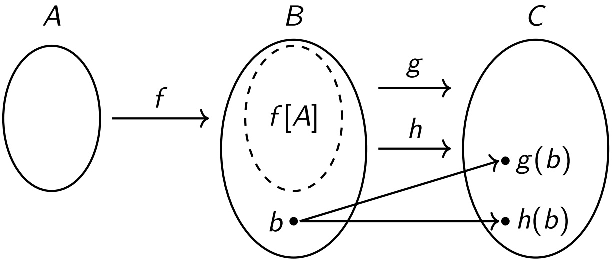
Theorem (Blok and Hoogland 2001)

Let K *be a quasivariety that is the equivalent algebraic semantics of a propositional logic* L *in the sense of Blok and Pigozzi. Then*

L *has the Beth definability property iff* K *has the weak ES property.*

Definition

K has the **strong ES property** if for every homomorphism $f\colon A \to B$ in K and $b \in B - f[A]$ there are $g, h\colon B \to C$ with $C \in K$ such that $gf = hf$ and $g(b) \neq h(b)$.



A homomorphism $f\colon A \to B$ in a class of algebras K is called a **monomorphism** if it is left cancellable; i.e., for every $g, h\colon C \to A$ with $C \in \mathsf{K}$

$$fg = fh \text{ implies } g = h.$$

In quasivarieties, the strong ES propery holds iff every monomorphism in K is regular (i.e., every injective homomorphism is the equalizer of a pair of maps) and it implies that every epimorphism is surjective (regular).

Theorem (Blok and Hoogland 2001)

Let K *be a quasivariety that is the equivalent algebraic semantics of a propositional logic* L *in the sense of Blok and Pigozzi. Then*

L *has the projective Beth definability property iff* K *has the strong ES.*

The strong ES property is the algebraic counterpart of the projective Beth property.

The surjectivity of any epimorphism in K corresponds to an intermediate definability property called the **infinite Beth definability property**.

The singleton/infinite/projective Beth definability properties have been extensively studied in nonclassical propositional logics.

- **Every** superintuitionistic propositional logic has the **Beth definability property** (Kreisel 1960).
- **Infinitely many** superintuitionistic propositional logics have the **infinite Beth definability property** (Bezhanishvili, Moraschini, Raftery 2017) and **infinitely many do not** (Moraschini, Wannenburg 2020).
- **16** superintuitionistic propositional logics have the **projective Beth definability property** (Maksimova 2000).

Implicit operations

L. Carai, M. Kurtzhals, and T. Moraschini, *The theory of implicit operations* (2025). Available online at arXiv:2512.14326

Implicit operations are partial functions whose values are implicitly defined and are preserved by homomorphisms.

Definition

An n-ary **partial function** on a class of algebras K is a family $f = \langle f^A : A \in \mathsf{K} \rangle$ of functions $f^A\colon \mathrm{dom}(f^A) \to A$ with $\mathrm{dom}(f^A) \subseteq A^n$.
A partial function f on K is said to be

- **implicit** when it is defined by a first-order formula $\varphi(x_1, \dots, x_n, y)$ in the language of K; i.e.,

  $$\mathrm{dom}(f^A) = \{\langle a_1, \dots, a_n \rangle \in A^n : \exists b \in A \text{ s.t. } A \vDash \varphi(a_1, \dots, a_n, b)\}$$

  and $f^A(a_1, \dots, a_n)$ is the unique element $b \in A$ such that $A \vDash \varphi(a_1, \dots, a_n, b)$;
- **an operation** when it is preserved by homomorphisms: for every homomorphism $h\colon A \to B$ with $A, B \in \mathsf{K}$ and $\langle a_1, \dots, a_n \rangle \in \mathrm{dom}(f^A)$ we have $\langle h(a_1), \dots, h(a_n) \rangle \in \mathrm{dom}(f^B)$ and

  $$h(f^A(a_1, \dots, a_n)) = f^B(h(a_1), \dots, h(a_n)).$$

Example

Complementation is an implicit operation of the variety BDL of bounded distributive lattices, which is implicitly defined by the formula $\varphi(x, y)$:

$$(x \wedge y = 0) \,\&\, (x \vee y = 1).$$

If $A \in \mathrm{BDL}$, then $A \vDash \varphi(a, b)$ if and only if b is a complement of a in A. It is an operation because complements are preserved by bounded lattice homomorphisms.

Example

Taking inverses is an implicit operation of the variety of monoids, which is implicitly defined by the formula

$$(xy = 1) \,\&\, (yx = 1).$$

Definition

A **primitive positive formula** (*pp formula* for short) is of the form

$$\exists x_1, \ldots, x_n \psi,$$

where ψ is a conjunction of equations.

The class of implicit operations of K defined by pp formulas is denoted by $\mathsf{imp}_{\mathrm{PP}}(\mathsf{K})$.

Theorem

Let f be an implicit operation on an elementary class K*. Then there exist* $f_1, \ldots, f_n \in \mathsf{imp}_{\mathrm{PP}}(\mathsf{K})$ *such that for each* $A \in \mathsf{K}$,

$$f^A = f_1^A \cup \cdots \cup f_n^A.$$

It is enough to work with implicit operations defined by pp formulas because they are the building blocks of implicit operations.

Definition

A class of algebras K has the **strong Beth definability property** when for each n-ary implicit operation f of K there exists a set $\mathcal{T}$ of n-ary terms in the language of K that interpolates f; i.e., for all $A \in \mathsf{K}$ and $\langle a_1, \ldots, a_n \rangle \in \mathrm{dom}(f^A)$ there exists $t \in \mathcal{T}$ such that

$$f^A(a_1, \ldots, a_n) = t(a_1, \ldots, a_n).$$

The strong Beth definability property holds iff every implicit operation can be obtained via compositions of basic operations.

Theorem

- *A quasivariety has the strong Beth definability property iff every* $f \in \mathrm{imp}_{\mathrm{PP}}(\mathsf{K})$ *can be interpolated by a single term.*
- *A universal class has the strong Beth definability property iff it has the strong ES property.*

Example

- The variety of bounded distributive lattices doesn't have the strong Beth definability property (complements are not expressible by terms).
- The variety of (commutative) monoids doesn't have the strong Beth definability property (inverses are not expressible by terms).

Example

- The variety of Boolean algebras has the strong Beth definability property.
- The variety of (abelian) groups has the strong Beth definability property.

The strong Beth definability property has applications in automated reasoning and formal verification.

Interpolation plays an important role in these contexts because **interpolants** can be thought of as **overapproximations** of a formula relative to another formula.

Several verification problems are formalized by representing set of states and transitions as quantifier-free formulas, so **quantifier-free interpolants** are particularly important.

Theorem (Bruttomesso, Ghilardi, Ranise (2014))

Under certain assumptions, if T_1, T_2 are two theories with quantifier-free interpolation and "the strong Beth definability property", then so does $T_1 \cup T_2$.

Beth companions: making implicit operations explicit

Definition

An implicit operation f of a universal class K is said to be **extendable** when for every $A \in \mathsf{K}$ there exists $B \in \mathsf{K}$ with $A \leq B$ such that f^{B} is total.

The set of extendable implicit operations defined by pp formulas is denoted by $\mathrm{ext}_{\mathrm{PP}}(\mathsf{K})$.

Example

- Complementation **is** an extendable operation in the variety of bounded distributive lattices.
- Taking inverses **is not** extendable in the variety of (commutative) monoids.

Let K be a class of algebras in the language $\mathscr{L}$ and $\mathcal{F} \subseteq \mathrm{ext}_{\mathrm{PP}}(\mathsf{K})$.

- $\mathscr{L}_{\mathcal{F}}$ denotes an expansion of $\mathscr{L}$ with a new function symbol for each $f \in \mathcal{F}$.
- We denote

$$\mathsf{K}[\mathscr{L}_{\mathcal{F}}] = \{A : A \in \mathsf{K} \text{ and } f^{A} \text{ is total for each } f \in \mathcal{F}\}$$

  and think of it as a class of algebra in the language $\mathscr{L}_{\mathcal{F}}$.

Definition

Let K and M be classes of algebras. Then M is a **Beth companion** of K when

- $\mathsf{M} = \mathbb{S}(\mathsf{K}[\mathscr{L}_{\mathcal{F}}])$ for some $\mathcal{F} \subseteq \mathrm{ext}_{\mathrm{PP}}(\mathsf{K})$,
- M has the strong Beth definability property.

The Beth companion completes a class into one with the strong Beth definability property.

Example

The variety of Boolean algebras is the Beth companion of the variety of bounded distributive lattices.

Theorem

Let $\mathsf{M} = \mathbb{S}(\mathsf{K}[\mathscr{L}_{\mathcal{F}}])$ *be a Beth companion of* K.

- *if* K *is a universal class, then* M *is a universal class;*
- *if* K *is a quasivariety, then* M *is a quasivariety;*
- *if* K *is a variety and each* $f \in \mathcal{F}$ *is defined by a conjunction of equations, then* M *is a variety.*

Theorem

If M *and* M' *are two Beth companions of a quasivariety* K, *then there exists a term equivalence between* M *and* M' *that fixes the language of* K.

The Beth companion of a quasivariety is essentially unique.

Idea of the proof:

$$\mathsf{imp}_{\mathrm{PP}}(\mathsf{K}) \quad \longleftrightarrow \quad \text{terms of } \mathsf{M}$$

$$\mathsf{imp}_{\mathrm{PP}}(\mathsf{K}) \quad \longleftrightarrow \quad \text{terms of } \mathsf{M}'$$

Class	**Beth companion**
Hilbert algebras	implicative semilattices
pseudocomplemented distributive lattices	Heyting algebras of depth ≤ 2
variety V generated by a Heyting algebra of depth ≤ 2	either V or V "plus certain modal operators"
variety V generated by a linearly ordered Heyting algebra A	none if $5 \leq \lvert A \rvert < \omega$ and V otherwise
MV-algebras	MV-algebras with division
variety V generated by a finite MV-chain of size n	V "plus a constant for $\frac{1}{n}$"

Class	**Beth companion**
(commutative) semigroups	none
(commutative) monoids	none
cancellative commutative semigroups	Abelian groups
cancellative commutative monoids	Abelian groups
torsion-free Abelian groups	Abelian groups with division
Abelian ℓ-groups	Abelian ℓ-groups with division

A variety of commutative monoids is said to be **inverse** if it validates the equation

$$x^n = x$$

for some $n \geq 2$.

Theorem

A variety of commutative monoids has a Beth companion if and only if it is inverse, in which case it is its own Beth companion.

L. Carai, M. Kurtzhals, and T. Moraschini, *Implicit operations in varieties of commutative monoids* (2026). Available online at arXiv:2603.13916

Definition

A Beth companion M of a class K is said to be

- **simple** when it is of the form $\mathsf{K}[\mathscr{L}_{\mathcal{F}}]$;
- **equational** when it is induced by some $\mathcal{F}$ whose members are defined by conjunctions of equations.

Theorem

Let M *be a Beth companion of a quasivariety* K. *Then the following holds:*

$$\mathsf{K} \textit{ has the AP} \Longrightarrow \mathsf{M} \textit{ is equational} \Longrightarrow \mathsf{M} \textit{ is simple.}$$

Moreover, if K *has the amalgamation property, so does* M.

Beth companions of quasivarieties with amalgamation are better behaved.

Theorem

Let K *be a relatively* ***congruence distributive quasivariety*** *whose class of relatively finitely subdirectly irreducible* ***(RFSI) members are closed under nontrivial subalgebras****. If* M *is a* ***simple Beth companion*** *of* K*, then*

- M *is a variety;*
- M *is arithmetical;*
- M *has the congruence extension property;*
- *The class of FSI members of* M *is closed under subalgebras.*

$$\begin{aligned} \text{quasivariety} &\longmapsto \text{variety} \\ \text{congruence distributivity} &\longmapsto \text{arithmeticity} \\ \emptyset &\longmapsto \text{CEP}. \end{aligned}$$

Example

Boolean algebras are the Beth companion of bounded distributive lattices. We gain **congruence permutability**.

A commutative ring is said to be **reduced** when it has no nonzero nilpotents. They form a relatively congruence distributive quasivariety RCR whose RFSI members are the integral domains.

Definition

Let ICM be the variety axiomatized by the equations

$$x = x^2x^*, \quad x = x^{**}, \quad (r_p(x))^p = (1 - p^*p)x$$

for each prime p, together with the axioms of commutative rings.

We call **implicitly closed meadows** the members of ICM.

Theorem

ICM *is a simple Beth companion of* RCR. *It is an arithmetic variety with the amalgamation property and the congruence extension property.*

Note that RCR does not have the AP nor the CEP.

L. Carai, M. Kurtzhals, and T. Moraschini, *A completion of reduced commutative rings* (2026). Available online at arXiv:2605.12661

Definition

Let K be a class of algebras and $A \leq B \in \mathsf{K}$. The **dominion** of A in B relative to K is the set

$$\begin{aligned}\mathrm{d}_{\mathsf{K}}(A, B) = \{ b \in B : {} & \text{for each pair of homomorphisms } g, h \colon B \to C \\ & \text{with } C \in \mathsf{K} \text{ and } g{\restriction}_A = h{\restriction}_A, \text{ we have } g(b) = h(b) \}.\end{aligned}$$

Proposition

Let K be a universal class. K has the strong ES property (and hence the strong Beth definability property) iff $\mathrm{d}_{\mathsf{K}}(A, B) = A$ for all $A \leq B \in \mathsf{K}$.

Theorem

Let K *be an elementary class. For every* $A \leq B \in \mathsf{K}$ *we have*

$$\begin{aligned}\mathrm{d}_{\mathsf{K}}(A, B) = \{ b \in B : {} & \textit{there exist } f \in \mathrm{imp}_{\mathrm{PP}}(\mathsf{K}) \textit{ and} \\ & \langle a_1, \ldots, a_n \rangle \in \mathrm{dom}(f^B) \cap A^n \textit{ such that } f^B(a_1, \ldots, a_n) = b \}.\end{aligned}$$

$\mathrm{d}_{\mathsf{K}}(A, B)$ is the subalgebra of B "generated by A with respect to the hidden structure of B".

Theorem

Let K *be a quasivariety with amalgamation, and* M *a Beth companion of* K*. For all* $A, B \in \mathsf{K}$ *and* $C \in \mathsf{M}$ *such that* $A \leq B \leq C{\restriction}_{\mathscr{L}_{\mathsf{K}}}$ *we have*

$$\mathsf{d}_{\mathsf{K}}(A, B) = \mathsf{Sg}^{C}(A) \cap B.$$

Beth companions can be used to compute dominions.

The variety of Abelian groups is the Beth companion of the quasivariety CCM of cancellative commutative monoids.

Example

Let $A, B \in \mathsf{CCM}$ and $G(B)$ be the Grothendieck group of B (i.e., the group of differences of B). Then $B \leq G(B){\restriction}_{\mathscr{L}_{\mathsf{CCM}}}$, and so

$$\mathrm{d}_{\mathsf{CCM}}(A, B) = \mathrm{Sg}^{G(B)}(A) \cap B = \{b \in B : \exists a_1, a_2 \in A \text{ s.t. } a_1 = b + a_2\}.$$

The variety of Boolean algebras is the Beth companion of the variety BDL of bounded distributive lattices.

Example

Let $A, B \in \mathsf{BDL}$ and $F(B)$ the free Boolean extension of B. Then $B \leq F(B){\restriction}_{\mathscr{L}_{\mathsf{BDL}}}$, and so

$$\mathrm{d}_{\mathsf{BDL}}(A, B) = \mathrm{Sg}^{F(B)}(A) \cap B = \text{join subsemilattice of } B \text{ generated by } \{b \in B : \exists a \in A \text{ s.t. } b \wedge a = 0 \text{ and } b \vee a \in A\}.$$

Definition

We say that $\Delta \subseteq \mathrm{imp}_{\mathrm{PP}}(\mathsf{K})$ is a **dominion base** for K when for all $A \leq B \in \mathsf{K}$ we have

$$\begin{aligned}\mathrm{d}_{\mathsf{K}}(A,B) = \{ & b \in B : \text{there exist } f \in \Delta \text{ and} \\ & \langle a_1, \dots, a_n \rangle \in \mathrm{dom}(f^B) \cap A^n \text{ such that } f^B(a_1, \dots, a_n) = b \}.\end{aligned}$$

Dominion bases are collections of implicit operations that are "general enough" to provide all the information about the hidden structure.

Theorem

Let K *be a quasivariety.* $\Delta \subseteq \mathrm{imp}_{\mathrm{PP}}(\mathsf{K})$ *is a dominion base iff for every* $f \in \mathrm{imp}_{\mathrm{PP}}(\mathsf{K})$ *there exist* $g \in \Delta$ *and terms* $t_1, \dots, t_m$ *of* K *such that* $f^A \subseteq g^A(t_1, \dots, t_m)$ *for each* $A \in \mathsf{K}$.

It would be useful to develop **anti-unification (generalization)** techniques to compute dominion bases.

Future research directions

- Understand the logical meaning of the (lack of a) Beth companion in classes of algebras providing algebraic semantics of logics.
- Investigate definability in first-order logics using algebraic tools.
- Study different versions of definability and analogs of Beth companions for them.
- Better understand the connection with interpolation.
- Generalize our results to languages with relation symbols (such as graphs) or to algebras with additional structure (such as topological groups).

THANK YOU!