

On the universal theory of the free pseudocomplemented distributive lattice

Luca Carai (University of Milan)

Joint work with Tommaso Moraschini

Kraków, 31 July 2026

TACL 2026

Pseudocomplemented distributive lattices and IPC^-

Definition

A **pseudocomplemented distributive lattice** P is a bounded distributive lattice equipped with a unary operation $\neg$ satisfying for any $a, b \in P$:

$$a \wedge b = 0 \quad \text{iff} \quad a \leq \neg b.$$

They are the $(\wedge, \vee, \neg, 0, 1)$ -subreducts of Heyting algebras.

The variety **PDL** provides an algebraic semantics for IPC^- , the fragment of IPC consisting of the propositional intuitionistic validities containing only the connectives $\wedge, \vee, \neg, \perp, \top$.

Theorem (Algebraic completeness of IPC^-)

Let t_φ be a term corresponding to a formula φ . Then

$$\text{PDL} \models t_\varphi = 1 \quad \text{iff} \quad \vdash_{\text{IPC}^-} \varphi.$$

Free pseudocomplemented distributive lattices

Since PDLs form a variety, for every cardinal κ there exists the **free PDL** $\mathbf{F}_{\text{PDL}}(\kappa)$ over κ generators.

$\mathbf{F}_{\text{PDL}}(\aleph_0)$ can be constructed by quotienting the set of all formulas in the language of IPC^- by setting two formulas φ and ψ equivalent iff $\vdash_{\text{IPC}} \varphi \leftrightarrow \psi$.

In particular, $\vdash_{\text{IPC}^-} \varphi$ iff the equivalence class of φ is the top of $\mathbf{F}_{\text{PDL}}(\aleph_0)$.

Our goals

- Determine whether the universal theory of $\mathbf{F}_{\text{PDL}}(\aleph_0)$ is decidable.
- Provide an axiomatization of the universal theory of $\mathbf{F}_{\text{PDL}}(\aleph_0)$.

Admissible rules

A **multiconclusion rule** is an expression of the form $\Gamma \Rightarrow \Delta$, where Γ, Δ are finite sets of formulas.

The expression $\Gamma \Rightarrow \Delta$ should be read as “if every formula in Γ holds, then some formula in Δ holds”.

Definition

We say that a rule $\Gamma \Rightarrow \Delta$ is **admissible** in a logic L if for every substitution σ we have that:

$\vdash_L \sigma(\gamma)$ for every $\gamma \in \Gamma$, then there exists $\delta \in \Delta$ such that $\vdash_L \sigma(\delta)$.

Free PDLs and admissible rules

Theorem

A rule $\Gamma \Rightarrow \Delta$ is admissible in IPC^- iff the universal first-order sentence

$$\forall \overline{x} \left((t_{\gamma_1} = 1 \ \& \ \cdots \ \& \ t_{\gamma_n} = 1) \Rightarrow (t_{\delta_1} = 1 \ \text{or} \ \cdots \ \text{or} \ t_{\delta_m} = 1) \right)$$

holds in $\mathbf{F}_{\text{PDL}}(\aleph_0)$.

For example, the rule $\{p \vee q\} \Rightarrow \{p, q\}$ is admissible in IPC^- as it corresponds to the sentence

$$\forall x, y \ (x \vee y = 1 \Rightarrow (x = 1 \ \text{or} \ y = 1)),$$

which holds in $\mathbf{F}_{\text{PDL}}(\aleph_0)$ (because the top element of $\mathbf{F}_{\text{PDL}}(\aleph_0)$ is join irreducible).

$\mathbf{F}_{\text{PDL}}(\aleph_0)$ and admissible rules

Rules for IPC^- correspond to universal first-order sentences of the form

$$\forall \bar{x} \left((t_{\gamma_1} = 1 \ \& \ \cdots \ \& \ t_{\gamma_n} = 1) \Rightarrow (t_{\delta_1} = 1 \ \text{or} \ \cdots \ \text{or} \ t_{\delta_m} = 1) \right),$$

while generic universal first-order sentences are conjunctions of

$$\forall \bar{x} \left((t_1 = t'_1 \ \& \ \cdots \ \& \ t_n = t'_n) \Rightarrow (s_1 = s'_1 \ \text{or} \ \cdots \ \text{or} \ s_m = s'_m) \right),$$

which are more general (IPC^- is **not** algebraizable in the sense of Blok and Pigozzi.).

The decidability of $\text{Th}_{\forall}(\mathbf{F}_{\text{PDL}}(\aleph_0))$ yields the decidability of admissibility in IPC^- . However, the axiomatization doesn't have to consist of universal sentences of the first kind (it won't), and so it doesn't correspond to a basis of admissible rules.

Strategy

- Use a duality for finite pseudocomplemented distributive lattices to describe the finite members of PDL that embed into $\mathbf{F}_{\text{PDL}}(\aleph_0)$.
- Exploit the local finiteness of PDL to obtain a description of the models of $\text{Th}_{\forall}(\mathbf{F}_{\text{PDL}}(\aleph_0))$; i.e., the members of the universal class $\mathbb{U}(\mathbf{F}_{\text{PDL}}(\aleph_0))$ generated by $\mathbf{F}_{\text{PDL}}(\aleph_0)$.
- Use the description of the members of $\mathbb{U}(\mathbf{F}_{\text{PDL}}(\aleph_0))$ to derive the decidability and the axiomatization.

Duality for finite PDL

A map $p: X \rightarrow Y$ between finite posets is said to be a **weak p-morphism** when it is order preserving and for all $x \in X$ and $y \in \max Y$,

if $p(x) \leq y$, there exists $z \in \max \uparrow x$ such that $p(z) = y$.

As a consequence of a duality for PDL due to Priestley (1975) we obtain.

Theorem

*The category of **finite pseudocomplemented distributive lattices** is dually equivalent to the category of **finite posets and weak p-morphisms**.*

finite posets

finite PDL

$$\begin{array}{ccc} X & \longrightarrow & (\text{Up}(X), \subseteq) \\ (\text{Jirr}(A), \geq) & \longleftarrow & A \end{array}$$

Duals of free finitely generated PDL

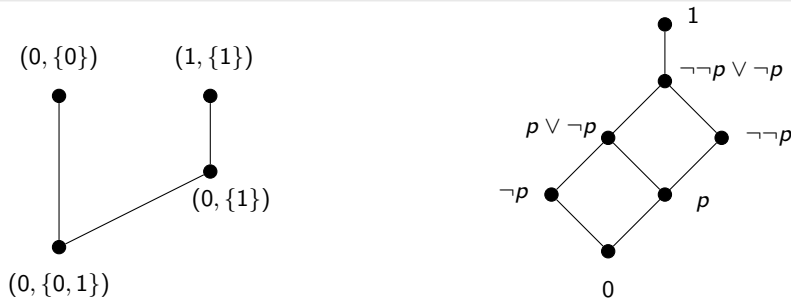
Theorem (Urquhart 1973 (see also Davey & Goldberg 1980))

The dual of the free 1-generated pseudocomplemented distributive lattice $\mathbf{F}_{\text{PDL}}(1)$ is the poset $P(1)$ with universe

$$\{(x, C) \in 2 \times \wp(2) : \emptyset \neq C \subseteq \uparrow x\},$$

ordered as follows:

$$(x, C) \leq (y, D) \iff x \leq y \text{ and } C \supseteq D.$$



Duals of free finitely generated PDL

Theorem (Urquhart 1973 (see also Davey & Goldberg 1980))

The dual of the free n -generated pseudocomplemented distributive lattice $\mathbf{F}_{\text{PDL}}(n)$ is the poset $P(n)$ with universe

$$\{(x, C) \in 2^n \times \wp(2^n) : \emptyset \neq C \subseteq \uparrow x\},$$

ordered as follows:

$$(x, C) \leq (y, D) \iff x \leq y \text{ and } C \supseteq D.$$

Posets with free skeleton

Theorem (C. & Moraschini 2025)

Let A be a finite PDL. Then A embeds into $\mathbf{F}_{\text{PDL}}(\aleph_0)$ if and only if its dual poset has a free skeleton.

A poset X with minimum $\perp$ is said to have a **free skeleton** when the following hold:

- for all $x \in X$ and nonempty $Y \subseteq \max \uparrow x$ there exists an element $s_{x,Y} \in \uparrow x$ such that

$$Y = \max \uparrow s_{x,Y};$$

- for all $x \in X$ and nonempty $Y, Z \subseteq \max \uparrow x$,

$$Y \subseteq Z \text{ implies } s_{x,Z} \leq s_{x,Y};$$

- for all $x \in X$ and nonempty $Y \subseteq \max X$,

$$\max \uparrow x \subseteq Y \text{ implies } s_{\perp,Y} \leq x.$$

Posets with free skeleton and free PDL

Theorem (C. & Moraschini 2025)

Let A be a finite PDL. Then A embeds into $\mathbf{F}_{\text{PDL}}(\aleph_0)$ if and only if its dual poset has a free skeleton.

Sketch of the proof: Let $A \in \text{PDL}$ be finite and X its dual poset.

- A embeds into $\mathbf{F}_{\text{PDL}}(\aleph_0)$ iff it embeds into $\mathbf{F}_{\text{PDL}}(n)$ for some n .
- A embeds into $\mathbf{F}_{\text{PDL}}(n)$ iff X is a weak p-morphic image of $P(n)$.
- $P(n)$ has a free skeleton.
- Onto weak p-morphisms $P(n) \rightarrow X$ transfer the free skeleton structure of $P(n)$ to X .
- If a finite poset X has a free skeleton then you can build an onto weak p-morphism $P(n) \rightarrow X$ for some n (this is the hard part).

A useful universal algebraic fact

Recall that for a class of algebras K the class of models of $\text{Th}_V(K)$ is $\mathbb{U}(K) = \text{ISP}_u(K)$.

Theorem

Let V be a locally finite variety and $K \subseteq V$. Then

$$\mathbb{U}(K) = \{A \in V : B \in \text{IS}(K) \text{ for every finite subalgebra } B \text{ of } A\}.$$

Main ingredient for the proof: each algebra A embeds into an ultraproduct of its finitely generated subalgebras.

Models of $\text{Th}_{\forall}(\mathbf{F}_{\text{PDL}}(\aleph_0))$

Theorem (C. & Moraschini 2025)

$$\mathbb{U}(\mathbf{F}_{\text{PDL}}(\aleph_0)) = \{A \in \text{PDL} : \text{duals of all finite subalgebras of } A \text{ have a free skeleton}\}.$$

It is well known that for all varieties V and infinite cardinal κ we have

$$\mathbb{U}(\mathbf{F}_V(\aleph_0)) = \mathbb{U}(\mathbf{F}_V(\kappa)) = \mathbb{U}(\{\mathbf{F}_V(n) : n \in \mathbb{Z}^+\}).$$

Therefore, we also obtain a characterization of the members of $\mathbb{U}(\mathbf{F}_{\text{PDL}}(\kappa))$ for every infinite cardinal κ and of $\mathbb{U}(\{\mathbf{F}_{\text{PDL}}(n) : n \in \mathbb{Z}^+\})$.

Axiomatization of $\text{Th}_{\forall}(\mathbf{F}_{\text{PDL}}(\aleph_0))$

The **atomic diagram** of a finite pseudocomplemented distributive lattice $A = \{a_1, \dots, a_n\}$ is the set of equations in the variables $x_1, \dots, x_n$

$$\{f(x_{i_1}, \dots, x_{i_m}) \approx x_k : f \in \{\wedge, \vee, \neg, 0, 1\} \text{ and } f^A(a_{i_1}, \dots, a_{i_m}) = a_k\};$$

together with the negated equations

$$\{x_i \not\approx x_j : i \neq j\}.$$

Theorem (C. & Moraschini 2025)

The theory $\text{Th}_{\forall}(\mathbf{F}_{\text{PDL}}(\aleph_0))$ is recursively axiomatizable by

$$\Sigma \cup \{\neg \exists x_1, \dots, x_n \bigwedge \text{diag}(A) : A \in \text{PDL is finite}$$

and its dual lacks a free skeleton\},

where Σ is a finite set of axioms of PDL.

We have also obtained an alternative axiomatization that, although still infinite, captures the idea of having a free skeleton in a more concrete way.

Decidability

Theorem (C. & Moraschini 2025)

$\text{Th}_{\forall}(\mathbf{F}_{\text{PDL}}(\aleph_0))$ is decidable.

Sketch of the proof:

- We have obtained a recursive axiomatization of $\text{Th}_{\forall}(\mathbf{F}_{\text{PDL}}(\aleph_0))$.
- Let V be a finitely axiomatizable and locally finite variety of finite type. If $\text{Th}_{\forall}(\mathbf{F}_V(\aleph_0))$ is recursively axiomatizable, then it is also decidable.
- We conclude that $\text{Th}_{\forall}(\mathbf{F}_{\text{PDL}}(\aleph_0))$ is decidable.

Corollary

Admissibility of multiconclusion rules in IPC^- is decidable.

Corollary

$\text{Th}_{\forall}(\mathbf{F}_{\text{PDL}}(\kappa))$ for κ infinite, and $\text{Th}_{\forall}(\{\mathbf{F}_{\text{PDL}}(n) : n \in \mathbb{Z}^+\})$ are decidable.

Idziak in 1987 showed that the elementary theory of $\{\mathbf{F}_{\text{PDL}}(n) : n \in \mathbb{Z}^+\}$ is undecidable.

Derivable and admissible rules in IPC^-

A single-conclusion rule $\Gamma \Rightarrow \delta$ is admissible in IPC^- if for every substitution σ we have that:

$$\vdash_{\text{IPC}^-} \sigma(\gamma) \text{ for every } \gamma \in \Gamma, \text{ we have } \vdash_{\text{IPC}^-} \sigma(\delta).$$

IPC^- can be defined also as a **consequence relation** (as a fragment of IPC).

Theorem (Mints 1976)

IPC^- is **structurally complete**: any admissible $\Gamma \Rightarrow \delta$ in IPC^- is derivable; i.e., $\Gamma \vdash_{\text{IPC}^-} \delta$.

Algebraically: for Φ “special quasiequation”, $\text{PDL} \models \Phi$ iff $\mathbf{F}_{\text{PDL}}(\aleph_0) \models \Phi$.

Nonetheless, $\text{PDL} \neq \mathbb{Q}(\mathbf{F}_{\text{PDL}}(\aleph_0))$ (some finite SI members of PDL don't have a free skeleton).

THANK YOU!