The Beth companion: making implicit operations explicit

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Implicit operations

An **implicit operation** in a class of algebraic structures consists of a partial function on each member of the class that **"behave like an operation"** and **are implicitly defined by a formula**.

Example

Complementation is an implicit operation in the class of **bounded distributive lattices**.

$\neg a$ is the unique element, when it exists, such that

$$a \wedge \neg a = 0 \quad \text{and} \quad a \vee \neg a = 1.$$

So, it is defined by the conjunction of equations

$$(x \wedge y = 0) \,\&\, (x \vee y = 1).$$

Moreover, complements are preserved by bounded lattice homomorphisms.

An **implicit operation** in a class of algebraic structures consists of a partial function on each member of the class that **“behave like an operation”** and **are implicitly defined by a formula**.

Example

Taking inverses is an implicit operation of the class of **monoids**.

a^{-1} is the unique element, when it exists, such that

$$a \cdot a^{-1} = 1 \quad \text{and} \quad a^{-1} \cdot a = 1.$$

So, it is defined by the conjunction of equations

$$(xy = 1) \,\&\, (yx = 1).$$

Moreover, inverses are preserved by monoid homomorphisms.

Definition

An n-ary **partial function** f on a class of algebras K is a family $\langle f^A : A \in \mathsf{K}\rangle$ of functions $f^A \colon \mathrm{dom}(f^A) \to A$ with $\mathrm{dom}(f^A) \subseteq A^n$.

A partial function f on K is said to be

- **implicit** when it is defined by a first-order formula $\varphi(x_1, \dots, x_n, y)$ in the language of K; i.e., for all $A \in \mathsf{K}$ and $a_1, \dots, a_n, b \in A$ we have $A \vDash \varphi(a_1, \dots, a_n, b)$ iff $\langle a_1, \dots, a_n\rangle \in \mathrm{dom}(f^A)$ and $b = f^A(a_1, \dots, a_n)$;
- **an operation** when it is preserved by homomorphisms: for every homomorphism $h \colon A \to B$ with $A, B \in \mathsf{K}$ and $\langle a_1, \dots, a_n\rangle \in \mathrm{dom}(f^A)$ we have $\langle h(a_1), \dots, h(a_n)\rangle \in \mathrm{dom}(f^B)$ and

$$h(f^A(a_1, \dots, a_n)) = f^B(h(a_1), \dots, h(a_n)).$$

Definition

A **primitive positive formula** (*pp formula* for short) is of the form

$$\exists x_1, \dots, x_n \psi,$$

where ψ is a conjunction of equations.

The class of implicit operations of K defined by pp formulas is denoted by $\mathrm{imp}_{\mathrm{PP}}(\mathsf{K})$.

It is enough to work with implicit operations defined by pp formulas because they are the **building blocks** of implicit operations.

Theorem

Let f be an implicit operation on an elementary class K. Then there exist $f_1, \dots, f_n \in \mathrm{imp}_{\mathrm{PP}}(\mathsf{K})$ such that for each $A \in \mathsf{K}$,

$$f^A = f_1^A \cup \dots \cup f_n^A.$$

When implicit operations are explicit

The strong Beth definability property

Definition

A class of algebras K has the **strong Beth definability property** when for all n-ary implicit operations f of K, $A \in$ K, and $\langle a_1, \ldots, a_n \rangle \in \mathrm{dom}(f^A)$ there exists a term t in the language of K such that

$$f^A(a_1, \ldots, a_n) = t(a_1, \ldots, a_n).$$

The strong Beth definability property holds when the values of every implicit operation can be obtained by **composing basic operations**.

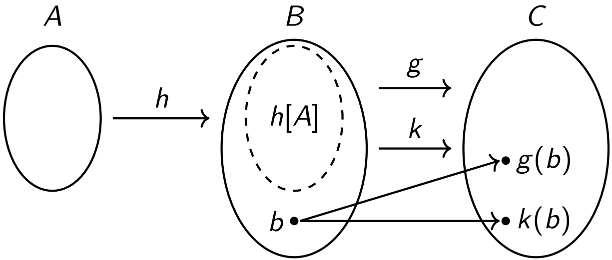
Proposition

Let K be a quasivariety with the strong Beth definability property and $f \in \mathrm{imp}_{\mathrm{PP}}(\mathsf{K})$. Then there exists a term t such that for all $A \in$ K and $\langle a_1, \ldots, a_n \rangle \in \mathrm{dom}(f^A)$ it holds that

$$f^A(a_1, \ldots, a_n) = t(a_1, \ldots, a_n).$$

Definition

A class K has the **strong ES property** if for all homomorphism $h\colon A \to B$ in K and $b \in B - h[A]$ there are $g, k\colon B \to C$ with $C \in K$ such that $gh = kh$ and $g(b) \neq k(b)$.



In a quasivariety: strong ES $\Leftrightarrow$ every mono is regular.

It implies that every epi is regular (surjective).

Theorem

A universal class has the strong Beth definability property iff it has the strong ES property.

Non-example

- The variety of bounded distributive lattices doesn't have the strong Beth definability property (complements are not expressible by terms).
- The variety of (commutative) monoids doesn't have the strong Beth definability property (inverses are not expressible by terms).

Example

- The variety of Boolean algebras has the strong Beth definability property.
- The variety of (Abelian) groups has the strong Beth definability property.

The Beth companion

A **Beth companion** of a class K is a canonical way of **making its implicit operations explicit**; i.e., forcing the strong Beth definability property.

Naive idea

Add all the implicit operations to the language and restrict to the algebras on which they are all total.

Problem: the resulting class would be trivial.

Better idea

Add **enough "well-behaved"** implicit operations to the language and restrict to the algebras on which they are all total.

Problem: the resulting class wouldn't be "unique" and would lack some desired structural properties.

Even better idea

Add **enough "well-behaved"** implicit operations to the language and restrict to the algebras on which they are all total. **Then also add their subalgebras in the expanded language**.

Definition

An implicit operation f of a universal class K is said to be **extendable** when for every $A \in \mathsf{K}$ there exists $B \in \mathsf{K}$ with $A \leq B$ such that f^B is total.

The set of extendable implicit operations defined by pp formulas is denoted by $\mathrm{ext}_{\mathrm{PP}}(\mathsf{K})$.

Example

- Complementation **is** an extendable operation in the variety of bounded distributive lattices.
- Taking inverses **is not** extendable in the variety of (commutative) monoids.

Let K be a class of algebras in the language $\mathscr{L}$ and $\mathcal{F} \subseteq \mathrm{ext}_{\mathrm{PP}}(\mathsf{K})$.

- $\mathscr{L}_{\mathcal{F}}$ denotes an expansion of $\mathscr{L}$ with a new function symbol for each $f \in \mathcal{F}$.
- We denote

$$\mathsf{K}[\mathscr{L}_{\mathcal{F}}] = \{A : A \in \mathsf{K} \text{ and } f^A \text{ is total for each } f \in \mathcal{F}\}$$

  and think of it as a class of algebra in the language $\mathscr{L}_{\mathcal{F}}$.

Definition

Let K be a class of algebras. Then a **Beth companion** of K is $\mathbb{S}(\mathsf{K}[\mathscr{L}_{\mathcal{F}}])$, for some $\mathcal{F} \subseteq \mathrm{ext}_{\mathrm{PP}}(\mathsf{K})$ that has the strong Beth definability property.

Example

The variety of Boolean algebras is the Beth companion of the variety of bounded distributive lattices.

The Beth companion of a quasivariety is **essentially unique**.

Theorem

If M *and* M' *are two Beth companions of a quasivariety* K*, then there exists a term equivalence between* M *and* M' *that fixes the language of* K*.*

Idea of the proof:

$$\mathsf{imp}_{\mathrm{PP}}(\mathsf{K}) \quad \longleftrightarrow \quad \text{terms of } \mathsf{M}$$

$$\mathsf{imp}_{\mathrm{PP}}(\mathsf{K}) \quad \longleftrightarrow \quad \text{terms of } \mathsf{M}'$$

Theorem

A Beth companion of a universal class (quasivariety) is a universal class (quasivariety).

Class	**Beth companion**
(commutative) monoids	none (Tommaso's talk)
cancellative commutative monoids	Abelian groups
torsion-free Abelian groups	Abelian groups with division
Abelian ℓ-groups	Abelian ℓ-groups with division
Reduced commutative rings	Miriam's talk
variety V generated by a Heyting algebra of depth ≤ 2	either V or V "plus certain modal operators"
variety V generated by a linearly ordered Heyting algebra A	none if $5 \leq \lvert A \rvert < \omega$ and V otherwise
Hilbert algebras	implicative semilattices
pseudocomplemented distributive lattices	Heyting algebras of depth ≤ 2
MV-algebras	MV-algebras with division
variety V generated by a finite MV-chain of size n	V "plus a constant for $\frac{1}{n}$"

Properties and applications of Beth companions

Definition

A Beth companion of a class K is said to be

- **equational** when it is induced by some $\mathcal{F}$ whose members are defined by conjunctions of equations;
- **simple** when it is of the form $\mathsf{K}[\mathscr{L}_{\mathcal{F}}]$;

Theorem

Let M *be a Beth companion of a quasivariety* K*. Then the following holds:*

$$\mathsf{K} \textit{ has the AP} \Longrightarrow \mathsf{M} \textit{ is equational} \Longrightarrow \mathsf{M} \textit{ is simple.}$$

Moreover, if K *has the amalgamation property, so does* M*.*

Theorem

A simple Beth companion of a variety is a variety.

Beth companions of quasivarieties with amalgamation are better behaved.

Definition

Let K be a quasivariety and $B \in \mathsf{K}$. A subalgebra A of B is **regular** (relative to K) if the inclusion $A \hookrightarrow B$ is a regular mono (i.e., an equalizer).

Regular subalgebras are the subalgebras closed under the implicit operations.

Definition

Let K be a quasivariety and $A \leq B \in \mathsf{K}$. The **dominion** $\mathrm{d}_{\mathsf{K}}(A, B)$ of A in B relative to K is the least regular subalgebra of B containing A.

Dominions measure the regularity of monomorphisms.

Theorem

Let K *be a quasivariety with the AP and* M *its Beth companion. If* $A, B \in \mathsf{K}$, $C \in \mathsf{M}$ *with* $A \leq B \leq C{\upharpoonright}_{\mathscr{L}_{\mathsf{K}}}$ *then*

$$\mathrm{d}_{\mathsf{K}}(A, B) = \mathrm{Sg}^{C}(A) \cap B.$$

Theorem

Let K *be a relatively* ***congruence distributive quasivariety*** *whose class of relatively finitely subdirectly irreducible* ***(RFSI) members are closed under nontrivial subalgebras****. If* M *is a* ***simple Beth companion*** *of* K*, then*

- M *is a variety;*
- M *is arithmetical;*
- M *has the congruence extension property;*
- *The class of FSI members of* M *is closed under subalgebras.*

$$\begin{aligned} \text{quasivariety} &\longmapsto \text{variety} \\ \text{congruence distributivity} &\longmapsto \text{arithmeticity} \\ \emptyset &\longmapsto \text{CEP}. \end{aligned}$$

Example

Boolean algebras are the Beth companion of bounded distributive lattices. We gain **congruence permutability**.

Future directions

- Understand the logical meaning of the (lack of a) Beth companion in classes of algebras providing algebraic semantics of logics.
- Study different versions of definability and analogs of Beth companions for them.
- Better understand the connection with the amalgamation property.
- Generalize our results beyond the algebraic setting.

Stay tuned

- Miriam's talk: *Implicit operations in reduced commutative rings*

  How to find the Beth companion of a class

- Tommaso's talk: *Implicit operations in varieties of commutative monoids*

  How to prove that a class doesn't admit a Beth companion

THANK YOU!

The theory of implicit operations

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`https://arxiv.org/abs/2512.14326`