

Baker-Beynon duality beyond semisimplicity

Luca Carai, University of Milan

Joint work with: S. Lapenta and L. Spada

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Baker-Beynon duality

Definition

A **Riesz space** V is an \mathbb{R} -vector space equipped with a lattice structure such that for every $u, v, w \in V$ and $0 \leq r \in \mathbb{R}$:

- if $u \leq v$, then $u + w \leq v + w$ and $ru \leq rv$.

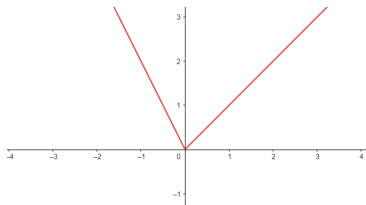
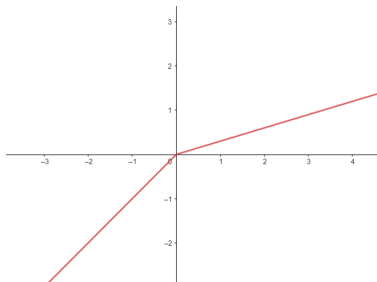
A map between Riesz spaces is a **Riesz space homomorphism** if it is a linear map and a lattice homomorphism.

Riesz spaces form a variety.

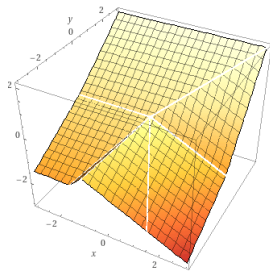
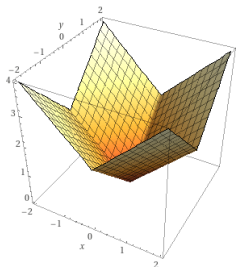
Examples of Riesz spaces

- \mathbb{R}
- \mathbb{R}^X for a set X
- $\mathbb{R} \overrightarrow{\times} \mathbb{R}$ (lexicographic product)
- $C(X, \mathbb{R})$ for a topological space X

A continuous function $f: \mathbb{R}^k \rightarrow \mathbb{R}$ is **piecewise linear (homogeneous)** if there exist $g_1, \dots, g_n: \mathbb{R}^k \rightarrow \mathbb{R}$ linear homogeneous functions (each in finitely many variables) such that for each $x \in \mathbb{R}^k$ we have $f(x) = g_i(x)$ for some $i = 1, \dots, n$.



A continuous function $f: \mathbb{R}^k \rightarrow \mathbb{R}$ is **piecewise linear (homogeneous)** if there exist $g_1, \dots, g_n: \mathbb{R}^k \rightarrow \mathbb{R}$ linear homogeneous functions (each in finitely many variables) such that for each $x \in \mathbb{R}^k$ we have $f(x) = g_i(x)$ for some $i = 1, \dots, n$.



The piecewise linear functions $f: \mathbb{R}^k \rightarrow \mathbb{R}$ form a Riesz space that we denote by **PWL(\mathbb{R}^k)**.

Theorem (Baker 1968)

Let κ be a cardinal. The free Riesz space on κ generators is isomorphic to $\text{PWL}(\mathbb{R}^\kappa)$. The free generators correspond to the projection maps onto each coordinate.

If $X \subseteq \mathbb{R}^\kappa$, we let $\text{PWL}(X) := \{f|_X \text{ with } f \in \text{PWL}(\mathbb{R}^\kappa)\}$.

Which Riesz spaces are isomorphic to $\text{PWL}(X)$ for some $X \subseteq \mathbb{R}^\kappa$?

Definition

An ℓ -ideal in a Riesz space is a subgroup (linear subspace) I that is convex, i.e. $|a| \leq |b|$ and $b \in I$ imply $a \in I$.

Definition

- A nontrivial Riesz space A is **simple** if $\{0\}$ and A are the only ℓ -ideals of A .
- A Riesz space is **semisimple** if the intersection of all its maximal ℓ -ideals is $\{0\}$.

Proposition

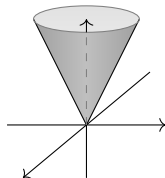
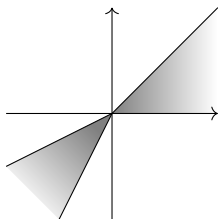
- *A Riesz space is simple iff it is isomorphic to \mathbb{R} .*
- *A Riesz space is semisimple iff it can be (subdirectly) embedded into a power of \mathbb{R} .*
- *$\text{PWL}(X)$ is semisimple for any $X \subseteq \mathbb{R}^\kappa$.*

Theorem (Baker 1968)

Every semisimple Riesz space is isomorphic to $\text{PWL}(C)$ for some closed cone $C \subseteq \mathbb{R}^k$.

Definition

A nonempty subset $C \subseteq \mathbb{R}^k$ is a **closed cone** if it is closed under multiplication by nonnegative scalars and it is topologically closed in \mathbb{R}^k with the euclidean topology.



This representation result extends to **Baker-Beynon duality**.

Let \mathcal{F}_κ be the free Riesz space over κ generators.

For any $T \subseteq \mathcal{F}_\kappa$ and $S \subseteq \mathbb{R}^\kappa$, we define the following operators.

$$V(T) = \{x \in \mathbb{R}^\kappa \mid t(x) = 0 \text{ for all } [t] \in T\}$$

$$I(S) = \{[t] \in \mathcal{F}_\kappa \mid t(x) = 0 \text{ for all } x \in S\}.$$

Galois connection

$$T \subseteq I(S) \quad \text{iff} \quad S \subseteq V(T).$$

- The fixpoints of VI are the closed cones of \mathbb{R}^κ .
- The fixpoints of IV are the ℓ -ideals of \mathcal{F}_κ that are intersections of maximal ℓ -ideals.

Proposition

V and I form a dual isomorphism between the poset of ℓ -ideals of \mathcal{F}_κ that are intersections of maximal ℓ -ideals and the poset of closed cones in \mathbb{R}^κ .

Theorem (Beynon 1974)

*The category of **semisimple Riesz spaces** and Riesz space homomorphisms is dually equivalent to the category of **closed cones in \mathbb{R}^κ** and piecewise linear maps between them.*

On objects:

Let A be a semisimple Riesz space, then $A \cong \mathcal{F}_\kappa / J$, where J is an intersection of maximal ℓ -ideals of \mathcal{F}_κ . Then map

$$A \mapsto V(J),$$

which is a closed cone in \mathbb{R}^κ .

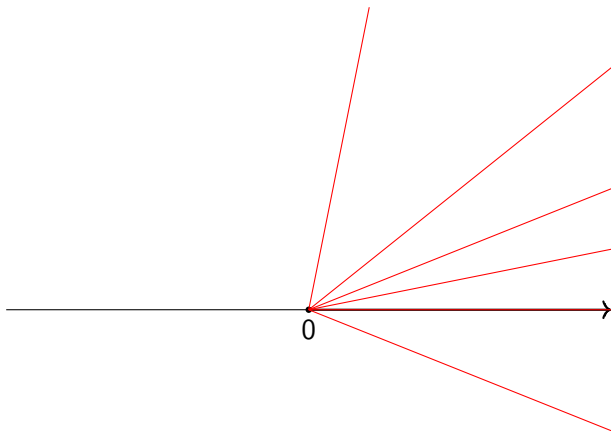
Let C be a closed cone in \mathbb{R}^κ . Then map

$$C \mapsto \text{PWL}(C),$$

which is semisimple and isomorphic to $\mathcal{F}_\kappa / I(C)$.

\mathbb{R} (as a Riesz space) is dual to the semiline $\{x \in \mathbb{R} \mid x \geq 0\}$.

Indeed, $\mathbb{R} \cong \text{PWL}(\{x \in \mathbb{R} \mid x \geq 0\})$. $\mathcal{F}_2 / \langle (x - y) \wedge y \wedge 0 \rangle$ is dual to $\{(x, y) \in \mathbb{R}^2 \mid 0 \leq y \leq x\}$.



Generalizing Baker-Beynon duality beyond semisimplicity

In the definition of the operators

$$V(T) = \{x \in \mathbb{R}^\kappa \mid t(x) = 0 \text{ for all } [t] \in T\} \text{ with } T \subseteq \mathcal{F}_\kappa$$

$$I(S) = \{[t] \in \mathcal{F}_\kappa \mid t(x) = 0 \text{ for all } x \in S\} \text{ with } S \subseteq \mathbb{R}^\kappa$$

we can replace \mathbb{R} with any Riesz space A and still get a Galois connection.

In the definition of the operators

$$V(T) = \{x \in A^\kappa \mid t(x) = 0 \text{ for all } [t] \in T\} \text{ with } T \subseteq \mathcal{F}_\kappa$$

$$I(S) = \{[t] \in \mathcal{F}_\kappa \mid t(x) = 0 \text{ for all } x \in S\} \text{ with } S \subseteq A^\kappa$$

we can replace \mathbb{R} with any Riesz space A and still get a Galois connection.

Caramello, Marra, and Spada (2021) observed that this can be done for any variety of algebras by replacing \mathbb{R} with any algebra in that variety.

They also show that this approach also works in a more categorical setting.

Our goal is to replace \mathbb{R} with a Riesz space that guarantees more ℓ -ideals of \mathcal{F}_κ to be fixpoints of IV .

It is not possible to obtain a Riesz space A such that for any κ the fixpoints of IV are all the ℓ -ideals of \mathcal{F}_κ .

Definition

An ℓ -ideal I is **prime** if $a \wedge b \in I$ implies $a \in I$ or $b \in I$.

Theorem

- A/I is linearly ordered iff I is prime.
- Every ℓ -ideal is an intersection of prime ℓ -ideals.
- Every Riesz space is subdirect product of linearly ordered ones.

Theorem (C., Lapenta, Spada)

Let α be a cardinal. There exists an ultrapower \mathcal{U} of \mathbb{R} in which all κ -generated (with $\kappa < \alpha$) linearly ordered Riesz spaces embed.

In particular, when $\alpha = \omega$ we can take \mathcal{U} to be any ultrapower of \mathbb{R} over a nonprincipal ultrafilter of a countably infinite set.

Fix a cardinal α and an ultrapower \mathcal{U} of \mathbb{R} in which all κ -generated with $\kappa < \alpha$ linearly ordered Riesz spaces embed. κ will denote an arbitrary cardinal smaller than α .

We consider the operators:

$$\begin{aligned} V(T) &= \{x \in \mathcal{U}^\kappa \mid t(x) = 0 \text{ for all } [t] \in T\} \text{ with } T \subseteq \mathcal{F}_\kappa \\ I(S) &= \{[t] \in \mathcal{F}_\kappa \mid t(x) = 0 \text{ for all } x \in S\} \text{ with } S \subseteq \mathcal{U}^\kappa. \end{aligned}$$

Galois connection

$$T \subseteq I(S) \quad \text{iff} \quad S \subseteq V(T).$$

- The fixpoints of IV are exactly the ℓ -ideals of \mathcal{F}_κ .
- We call $S \subseteq \mathcal{U}^\kappa$ such that $S = VI(S)$ a **generalized closed cone**.

Proposition

V and I establish a dual isomorphism between the poset of ℓ -ideals of \mathcal{F}_κ and the poset of **generalized closed cones** in \mathcal{U}^κ .

Definition

- We say that a map $\mathcal{U}^\kappa \rightarrow \mathcal{U}^\mu$ is **definable** if its components are defined by terms in the language of Riesz spaces.
- If $X \subseteq \mathcal{U}^\kappa$, we denote by **Def**(X) the set of definable maps $f: X \rightarrow \mathcal{U}$.

Theorem (C., Lapenta, Spada)

*The category of **Riesz spaces** with less than α generators and Riesz space homomorphisms is dually equivalent to the category of **generalized closed cones** in \mathcal{U}^κ for some $\kappa < \alpha$ and definable maps.*

On objects:

Let A be a κ -generated Riesz space, so $A \cong \mathcal{F}_\kappa / J$. Then map

$$A \mapsto V(J),$$

which is a generalized closed cone in \mathcal{U}^κ .

Let C be a generalized closed cone in \mathcal{U}^κ . Then map

$$C \mapsto \text{Def}(C),$$

which is isomorphic to $\mathcal{F}_\kappa / I(C)$.

Consequences and applications

Proposition

- The generalized closed cones in \mathcal{U}^κ (together with \emptyset) form the closed of a topology on \mathcal{U}^κ . The closure of a nonempty $X \subseteq \mathcal{U}^\kappa$ is $\vee I(X)$.
- \mathbb{R}^κ is a subset of \mathcal{U}^κ and the closed subsets of \mathbb{R}^κ with the subspace topology are exactly the closed cones (and \emptyset).

\mathcal{F}_κ	\mathbb{R}^κ (Baker-Beynon)	\mathcal{U}^κ (gen. Baker-Beynon)
maximal ℓ -ideals	half-lines from the origin	closures of points of \mathbb{R}^κ (except the origin)
intersections of maximal ℓ -ideals	closed cones	closures of nonempty subsets of \mathbb{R}^κ
prime ℓ -ideals		irreducible closed subsets = closures of points of \mathcal{U}^κ (except the origin)
ℓ -ideals		generalized closed cones

Definition

- Recall that a Riesz space is **semisimple** if the intersection of all its maximal ℓ -ideals is $\{0\}$.
- A Riesz space A is called **archimedean** if for every $a, b \in A$, we have that $na \leq b$ for all $n \in \mathbb{N}$ implies $a \leq 0$.
- Semisimplicity always implies archimedeanity.
- Archimedeanity implies semisimplicity in the presence of a strong order-unit (e.g., in the finitely generated setting).

Theorem

Let A be a Riesz space and $C \subseteq \mathcal{U}^\kappa$ its dual generalized closed cone. A is semisimple iff $C = \text{VI}(C \cap \mathbb{R}^\kappa)$, i.e. C is the closure of $C \cap \mathbb{R}^\kappa$ in \mathcal{U}^κ .

Note that $C \cap \mathbb{R}^\kappa$ is the closed cone in \mathbb{R}^κ corresponding to A under Baker-Beynon duality.

For any natural number $n \geq 1$ let $\pi_n: \mathcal{U}^\omega \rightarrow \mathcal{U}^n$ be the map that sends $(x_i)_{i \in \omega}$ to (x_1, \dots, x_n) .

Theorem

Let A be an ω -generated Riesz space and $C \subseteq \mathcal{U}^\omega$ its dual generalized closed cone.

Then A is archimedean iff

$$C = \bigcap_{n=1}^{\infty} \pi_n^{-1}[\text{VI}(\text{VI}(\pi_n[C]) \cap \mathbb{R}^n)],$$

where the subsets $\pi_n^{-1}[\text{VI}(\text{VI}(\pi_n[C]) \cap \mathbb{R}^n)]$ form a decreasing sequence of generalized closed cones in \mathcal{U}^ω .

When $\kappa > \omega$, the decreasing sequence must be replaced by a downdirected family of generalized closed cones in \mathcal{U}^κ .

Embedding $\mathrm{Spec}(\mathcal{F}_\kappa)$ into \mathcal{U}^κ

If A is a Riesz space, then $\text{Spec}(A) = \{\text{prime } \ell\text{-ideals of } A\}$ is called the **spectrum** of A . It is naturally equipped with the **Zariski topology** generated by the closed subsets $\{P \in \text{Spec}(A) \mid a \in P\}$, where a ranges in A .

If P is a prime ℓ -ideal of \mathcal{F}_κ , then $V(P)$ is the closure of a point of \mathcal{U}^κ . For each prime ℓ -ideal P choose one such point and denote it by $\mathcal{E}(P)$.

Theorem

- $\mathcal{E}: \text{Spec}(\mathcal{F}_\kappa) \rightarrow \mathcal{U}^\kappa$ is a topological embedding.
- The posets of the open subsets of $\text{Spec}(\mathcal{F}_\kappa)$ and of $\mathcal{U}^\kappa \setminus \{O\}$ are isomorphic.

$\mathcal{E}: \text{Spec}(\mathcal{F}_\kappa) \rightarrow \mathcal{U}^\kappa$ can be thought of as a **coordinatization** of $\text{Spec}(\mathcal{F}_\kappa)$ with coordinates in \mathcal{U} .

- By the correspondence theorem, if $A \cong \mathcal{F}_\kappa / J$, then we can think of $\text{Spec}(A)$ as a subspace of $\text{Spec}(\mathcal{F}_\kappa)$.
- \mathcal{E} restricts to an embedding of $\text{Spec}(A)$ into \mathcal{U}^κ whose image is $\mathcal{E}[\text{Spec}(\mathcal{F}_\kappa)] \cap V(J)$.

While the spectrum as a topological space is not sufficient to recover the original Riesz space, the coordinatization is enough:

Theorem

$A \cong \text{Def}(\mathcal{E}[\text{Spec}(A)])$ for any Riesz space A .

\mathcal{U} as the non-standard line

Let $\alpha = \omega$ and assume that \mathcal{U} is an ultrapower of \mathbb{R} defined as $\mathcal{U} = \mathbb{R}^{\mathbb{N}}/\mathcal{F}$ with \mathcal{F} a nonprincipal ultrafilter of $\mathcal{P}(\mathbb{N})$.

We can think of \mathcal{U} as a non-standard line. \mathcal{U} is a linearly ordered field containing (a copy of) \mathbb{R} . The elements of \mathcal{U} are called **hyperreal numbers**.

- $x \in \mathcal{U}$ is **infinitesimal** if $|x| \leq r$ for every $0 < r \in \mathbb{R}$.
- $x \in \mathcal{U}$ is **unlimited** if $|x| \geq r$ for every $0 < r \in \mathbb{R}$.

Working with an ultrapower allows us to define the **enlargement** ${}^*A \subseteq \mathcal{U}^n$ of a subset $A \subseteq \mathbb{R}^n$, as well as the enlargement ${}^*f: \mathcal{U}^n \rightarrow \mathcal{U}$ of a function $f: \mathbb{R}^n \rightarrow \mathbb{R}$.

Theorem (Transfer principle)

Let φ be a first-order sentence. Then φ is true in \mathbb{R} iff ${}^\varphi$ is true in \mathcal{U} .*

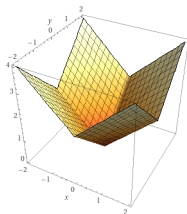
- $A \subseteq {}^*A$.
- If A is finite, then $A = {}^*A$.
- ${}^*\mathbb{Q}$ contains both nonzero infinitesimal and unlimited numbers.

Let $g: \mathcal{U}^n \rightarrow \mathcal{U}$ be definable, i.e. there is a term t such that $g(x) = t(x)$ for all $x \in \mathcal{U}^n$. If $f: \mathbb{R}^n \rightarrow \mathbb{R}$ is the piecewise linear function defined by the same term, then $g = {}^*f$.

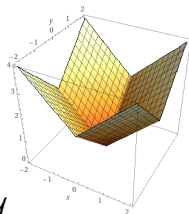
Proposition

Let $C \subseteq \mathcal{U}^n$ be a generalized closed cone. Then $\text{Def}(C) = \{({}^*f)|_C \mid f: \mathbb{R}^n \rightarrow \mathbb{R} \text{ piecewise linear}\}$.

The graph of ${}^*f: \mathcal{U}^n \rightarrow \mathcal{U}$ is just the enlargement of the graph of f .



$$f: \mathbb{R}^2 \rightarrow \mathbb{R}$$

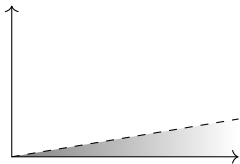


$${}^*f: \mathcal{U}^2 \rightarrow \mathcal{U}$$

Definable functions naturally generalize piecewise linear functions.

Let $\mathbb{R} \overrightarrow{\times} \mathbb{R}$. Then its dual generalized closed cone is

$$C = \{(x, y) \in \mathcal{U}^2 \mid x > 0, y \geq 0, \text{ and } y/x \text{ is infinitesimal}\} \cup \{(0, 0)\}.$$



So,

$$\begin{aligned} \mathbb{R} \overrightarrow{\times} \mathbb{R} &\cong \text{Def}(C) = \{(*f)|_C \mid f: \mathbb{R}^2 \rightarrow \mathbb{R} \text{ piecewise linear}\} \\ &= \{(*f)|_C \mid f: \mathbb{R}^2 \rightarrow \mathbb{R} \text{ linear}\}. \end{aligned}$$

Indexes and prime ideals

We have seen that prime ℓ -ideals of \mathcal{F}_n correspond to the closures of the points of \mathcal{U}^n .

We want to understand how these subsets of \mathcal{U}^n look like.

Theorem (Orthogonal decomposition)

If $x \in \mathcal{U}^n$, then $x = \alpha_1 v_1 + \cdots + \alpha_k v_k$ where $\alpha_1, \dots, \alpha_k \in \mathcal{U}$ are positive, α_{i+1}/α_i is infinitesimal for each $i < k$, and $v_1, \dots, v_k \in \mathbb{R}^n$ are orthonormal vectors. Furthermore, this decomposition is unique.

Definition

- We call a finite sequence (v_1, \dots, v_k) of orthonormal vectors in \mathbb{R}^n an **index**.
- We denote by $\iota(x)$ the index (v_1, \dots, v_k) made of the vectors appearing in the orthogonal decomposition of $x \in \mathcal{U}^n$.
- Let \mathbf{v}, \mathbf{w} be two indexes. We write $\mathbf{v} \leq \mathbf{w}$ when \mathbf{v} is a truncation of \mathbf{w} , i.e. $\mathbf{v} = (v_1, \dots, v_h)$ and $\mathbf{w} = (v_1, \dots, v_k)$ for $h \leq k$.

Definition

If \mathbf{v} is an index, let $\text{Cone}(\mathbf{v}) := \{y \in \mathcal{U}^n \mid \iota(y) \leq \mathbf{v}\}$.

Theorem

The closure of x in \mathcal{U}^n is $\text{Cone}(\iota(x))$.

Theorem

*$\iota \circ \text{Cone}: \mathbf{v} \mapsto \iota(\text{Cone}(\mathbf{v}))$ induces an order-isomorphism between the set of *indexes* ordered by truncation and $\text{Spec}(\mathcal{F}_n)$ ordered by reverse inclusion.*

If $\mathbf{v} = (v_1, \dots, v_k)$ is an index, then we call a subset of \mathbb{R}^n a **v-cone** if it is the positive span of $\{\sum_{i=1}^h r_i v_i \mid h \leq k\}$ for some $0 < r_1, \dots, r_k \in \mathbb{R}$.

Theorem

Cone(\mathbf{v}) is the intersection of the enlargements of all the \mathbf{v} -cones.

Corollary (Panti 1999)

Every prime ℓ -ideal of $\text{PWL}(\mathbb{R}^n)$ is of the form

$$\{f \in \text{PWL}(\mathbb{R}^n) \mid f \text{ vanishes on some } \mathbf{v}\text{-cone}\}$$

for some index \mathbf{v} .

Main idea of the proof: Let $f \in \text{PWL}(\mathbb{R}^n)$. By the transfer principle, $*f$ vanishes on $\text{Cone}(\mathbf{v})$ iff f vanishes on some \mathbf{v} -cone.

Recall: if we map $P \in \operatorname{Spec}(\mathcal{F}_n)$ to a point $x \in \mathcal{U}^n$ such that $V(P)$ is the closure of x , then we get an **embedding** $\mathcal{E}: \operatorname{Spec}(\mathcal{F}_n) \rightarrow \mathcal{U}^n$.

Indexes allow us to choose x for every P in a canonical way (modulo fixing a positive infinitesimal $\varepsilon \in \mathcal{U}$).

If $P \in \operatorname{Spec}(\mathcal{F}_n)$, then there is a unique index $\mathbf{v} = (v_1, \dots, v_k)$ such that $V(P) = \operatorname{Cone}(\mathbf{v})$. Define

$$\mathcal{E}(P) := v_1 + \varepsilon v_2 + \dots + \varepsilon^{k-1} v_k.$$

If P is a maximal ℓ -ideal, then the corresponding index consists of a single vector $\mathbf{v} = (v_1)$. Therefore, $\mathcal{E}(P) \in \mathbb{R}^n$.

We have $\mathcal{E}[\text{Spec}(\mathcal{F}_1)] = \{-1, 1\} \subseteq \mathcal{U}$.

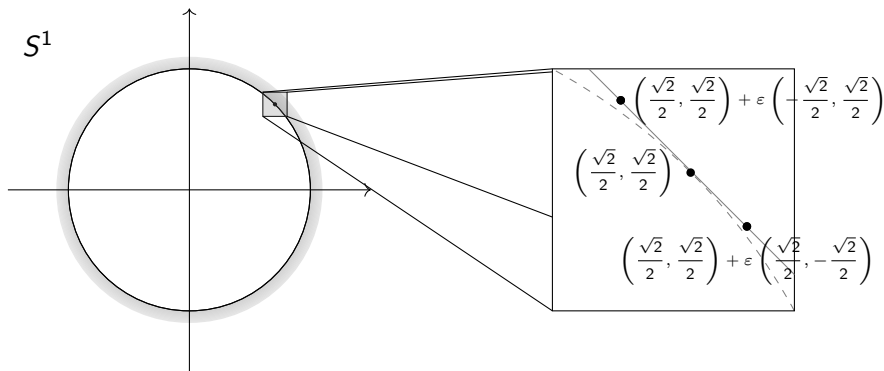


Note that $\text{Spec}(\mathcal{F}_1) = \text{MaxSpec}(\mathcal{F}_1)$.

$\text{Spec}(\mathcal{F}_2)$

We have $\mathcal{E}[\text{MaxSpec}(\mathcal{F}_2)] = S^1 \subseteq \mathbb{R}^2 \subseteq \mathcal{U}^2$. We have

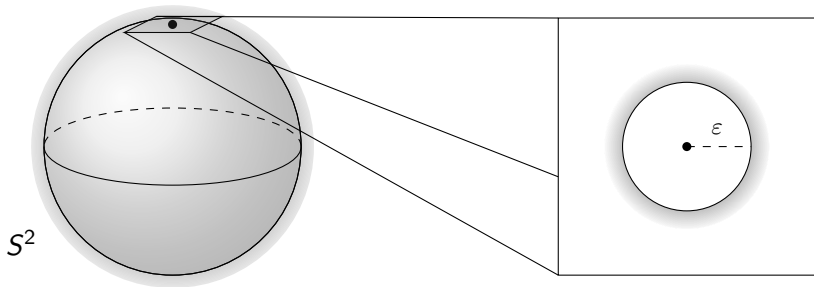
$\mathcal{E}[\text{Spec}(\mathcal{F}_2)] \subseteq \mathcal{U}^2$ consists of points infinitesimally close to S^1 .



$\mathrm{Spec}(\mathcal{F}_3)$

We have $\mathcal{E}[\mathrm{MaxSpec}(\mathcal{F}_3)] = S^2 \subseteq \mathbb{R}^3 \subseteq \mathcal{U}^3$. We have

$\mathcal{E}[\mathrm{Spec}(\mathcal{F}_3)] \subseteq \mathcal{U}^3$ consists of points infinitesimally close to S^2 .



- Extending Marra-Spada duality for semisimple MV-algebras and Riesz MV-algebras beyond semisimplicity:
 - $\mathcal{U} \rightarrow {}^*[0, 1]$.
 - Indexes: $(v_1, \dots, v_k) \rightarrow (x_0, v_1, \dots, v_k)$.
 - Irreducible closed subsets:
“infinitesimally wide cones” \rightarrow “infinitesimal simplexes”.
 - Describe dually the equivalence between MV-algebras and ℓ -groups with strong order-unit.
 - Describe dually the equivalence between perfect MV-algebras and ℓ -groups.
- Generalize the non-standard techniques and the indexes to the infinitely generated case.

მადლობა

(Thank you)