Free algebras and coproducts in varieties of Gödel algebras

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Heyting algebras provide the algebraic semantics for the intuitionistic propositional calculus IPC.

Definition

A Heyting algebra H is a distributive lattice equipped with a binary operation $\to$ satisfying

$$a \wedge b \leq c \quad \text{iff} \quad a \leq b \to c$$

for any $a, b, c \in H$.

Gödel algebras provide the algebraic semantics for the propositional Gödel-Dummett logic LC, which is obtained by adding the prelinearity axiom $(p \to q) \vee (q \to p)$ to IPC.

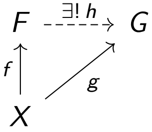
Definition

A Heyting algebra G is called a Gödel algebra if $(a \to b) \vee (b \to a) = 1$ for any $a, b \in G$.

We can think of LC as the extension of IPC in which "the truth values are linearly ordered". The variety of Gödel algebras is generated by the class of all (finite) Heyting chains.

LC is also the logic of $[0, 1]$ as a Heyting chain, and hence it can also be thought of as a fuzzy logic. In fact, LC is the t-norm fuzzy logic associated to the minimum t-norm.

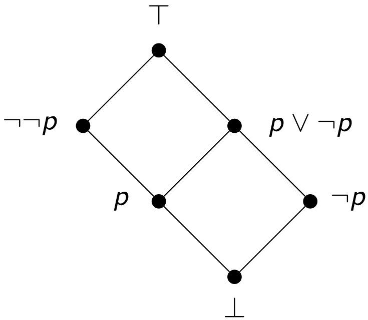
A Gödel algebra F is said to be free over a set X if there exists a map $f\colon X \to F$ such that for any Gödel algebra G and map $g\colon X \to G$, there is a unique Heyting homomorphism $h\colon F \to G$ with $g = h \circ f$.



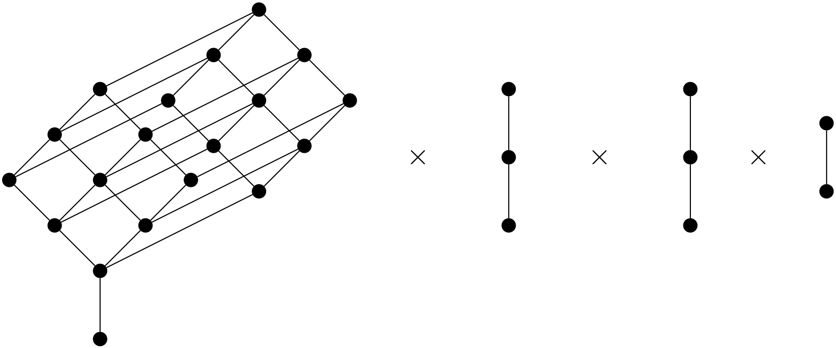
Free Gödel algebras arise as Lindenbaum-Tarski algebras for LC:

Let $\mathrm{Form}(X)$ be the set of formulas with variables from a set X and define $\varphi \sim \psi$ iff $\vdash_{\mathrm{LC}} \varphi \leftrightarrow \psi$. Then $\mathrm{Form}(X)/{\sim}$ is a Gödel algebra free over X.

Free Gödel algebra over 1-generator



Free Gödel algebra over 2-generators



It has 342 elements.

Definition

An Esakia root system is a Stone space X with a partial order $\leq$ such that:

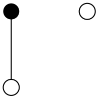
- if $x \in X$, then ${\uparrow}x$ is closed and a chain;
- if $V \subseteq X$ is clopen, then ${\downarrow}V$ is clopen.

Esakia root systems		Gödel algebras
X	$\longrightarrow$	$\mathrm{ClopUp}(X)$
$\mathrm{Spec}(G)$	$\longleftarrow$	G

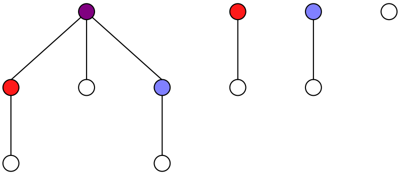
Theorem (Esakia duality for Gödel algebras)

The category of Gödel algebras and Heyting homomorphisms is dually equivalent to the category of Esakia root systems and continuous p-morphisms.

Dual of the free Gödel algebra over 1 generator



Dual of the free Gödel algebra over 2 generators



Proposition (Grigolia 1980s)

The dual of the free Gödel algebra over n generators is isomorphic to the set of all nonempty chains in $\mathcal{P}(\{1, \ldots, n\})$ ordered by

$$C \leq D \text{ iff } D \text{ is an upset of } C.$$

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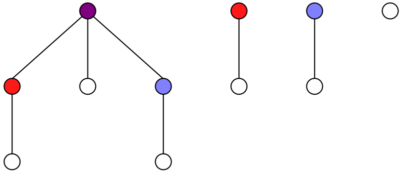

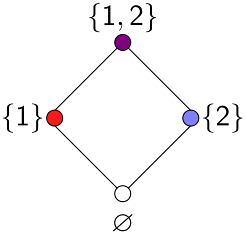

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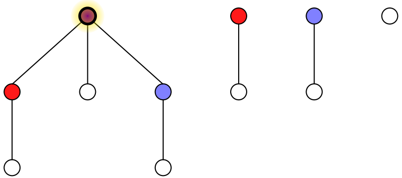

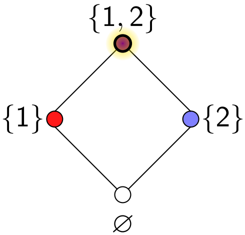

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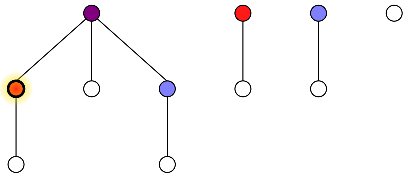

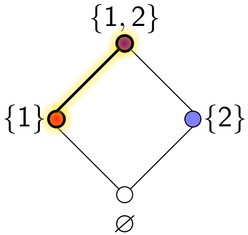

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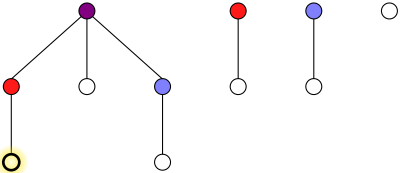

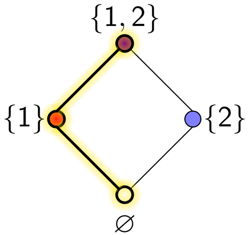

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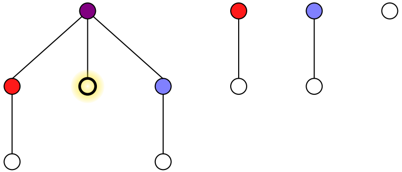

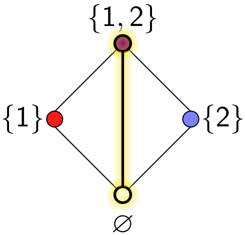

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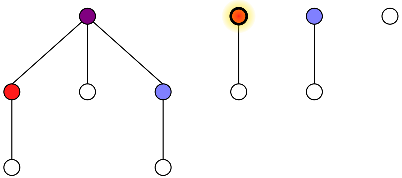

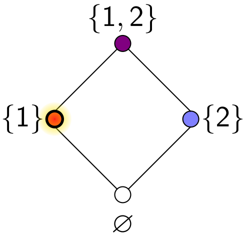

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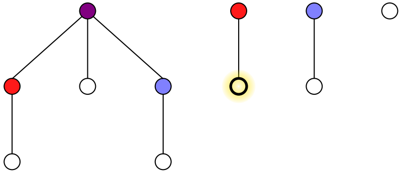

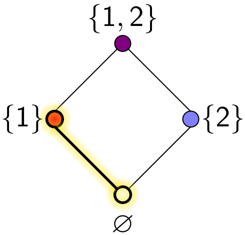

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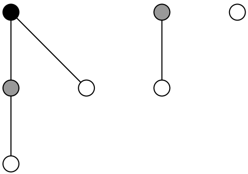
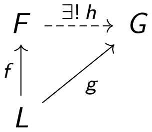


Which Gödel algebra is dual to what we get when we replace $\mathcal{P}(\{1,\ldots,n\})$ with an arbitrary finite poset P?

A Gödel algebra F is said to be free over a distributive lattice L if there exists a lattice homomorphism $f\colon L \to F$ such that for any Gödel algebra G and lattice homomorphism $g\colon L \to G$, there is a unique Heyting algebra homomorphism $h\colon F \to G$ with $g = h \circ f$.



Theorem (Aguzzoli, Gerla, and Marra 2008)

Let L be a finite distributive lattice and P a poset such that $L \cong \mathrm{Up}(P)$. The poset of all nonempty chains in P is the Esakia dual of the Gödel algebra free over L.

How to generalize this result to infinite distributive lattices?

Definition

A Priestley space is a Stone space X with a partial order $\leq$ such that

- $x \not\leq y$ implies there is U clopen upset such that $x \in U$ and $y \notin U$.

Priestley spaces		Distributive lattices
X	$\longrightarrow$	$\mathrm{ClopUp}(X)$
$\mathrm{Spec}(L)$	$\longleftarrow$	L

Theorem (Priestley duality 1972)

The category of distributive lattices and lattice homomorphisms is dually equivalent to the category of Priestley spaces and continuous order-preserving maps.

Definition

If X is a Priestley space, we define

$$\mathrm{CC}(X) := \{C \subseteq X \mid C \text{ is a nonempty closed chain}\}.$$

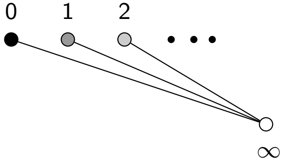
We order $\mathrm{CC}(X)$ by setting $C \leq D$ iff D is an upset of C.

We topologize $\mathrm{CC}(X)$ with the Vietoris topology, which is generated by the subbasis consisting of the sets $\Box V$ and $\Diamond V$ for any V clopen of X:

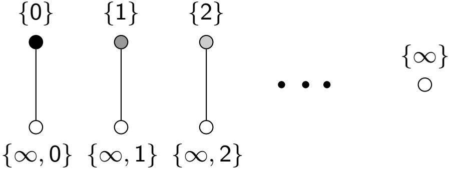
$$\Box V = \{C \in \mathrm{CC}(X) \mid C \subseteq V\},$$
$$\Diamond V = \{C \in \mathrm{CC}(X) \mid C \cap V \neq \varnothing\}.$$

Theorem (C. 2024)

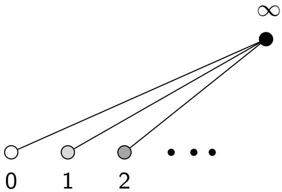
- *If X is a Priestley space, then $\mathrm{CC}(X)$ is an Esakia root system.*
- *Let L be a distributive lattice and X its Priestley dual. Then the Gödel algebra free over L is dual to the Esakia root system $\mathrm{CC}(X)$.*



X



$\mathrm{CC}(X)$



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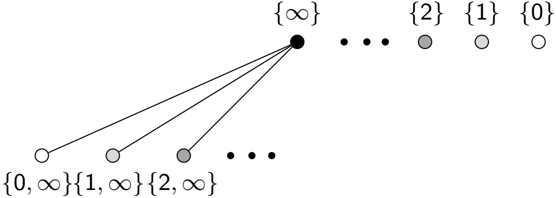


$\mathrm{CC}(X)$

Let $\mathbf{2}$ be the 2-element chain with the discrete topology. If S is a set, we consider $\mathbf{2}^S$ with the product topology and the product order.

Proposition

$\mathbf{2}^S$ is a Priestley space dual to the distributive lattice free over S.

Corollary (C. 2024)

The Gödel algebra free over a set S is dual to the Esakia root system $\mathrm{CC}(\mathbf{2}^S)$.

Ghilardi in 1992 showed that finitely generated free Heyting algebras are bi-Heyting algebras.

Definition

Let L be a distributive lattice.

- L is a co-Heyting algebra if its order dual is a Heyting algebra.
- L is a bi-Heyting algebra if it is both a Heyting and a co-Heyting algebra.

We can characterize when a Gödel algebra free over a distributive lattice is a bi-Heyting algebra.

Theorem (C. 2024)

Let F be the Gödel algebra free over a distributive lattice L. Then F a bi-Heyting algebra iff L is a co-Heyting algebra.

Since free distributive lattices are co-Heyting algebras:

Corollary (C. 2024)

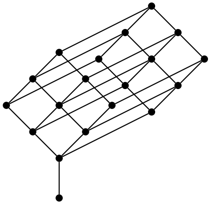
Free Gödel algebras are bi-Heyting algebras.

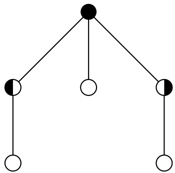
Coproducts of Gödel algebras

Coproducts of Heyting algebras (dually, products of Esakia spaces) are complicated. Already the coproduct of the 3-element Heyting chain with itself is infinite.

Coproducts of Gödel algebras (dually, products of Esakia root systems) are simpler. D'Antona and Marra in 2006 dually described binary coproducts of finite Gödel algebras, which are always finite.

The following are the coproduct of the 3-element Heyting chain with itself in the category of Gödel algebras and its dual.





Let $\{X_i\}$ be a family of Esakia root systems. We denote by $\prod_i X_i$ their cartesian product with the product topology and the product order.

Definition

Let $\bigotimes_i X_i$ be the subspace of $\mathrm{CC}(\prod_i X_i)$ given by

$$\bigotimes_i X_i := \{C \in \mathrm{CC}(\prod_i X_i) \mid \pi_i[C] \text{ is an upset of } X_i \text{ for each } i \in I\}.$$

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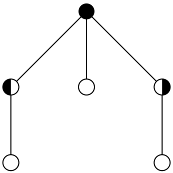

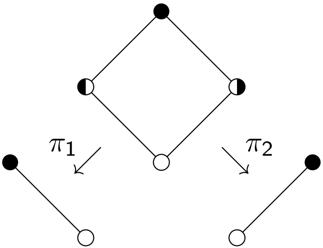

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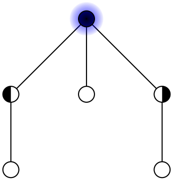

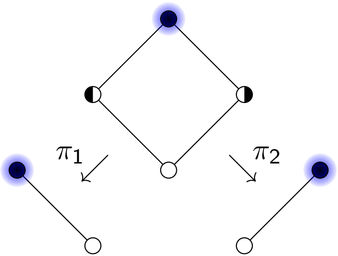

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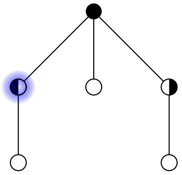

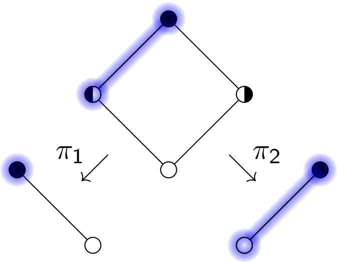

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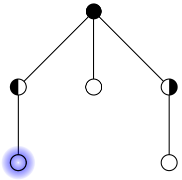

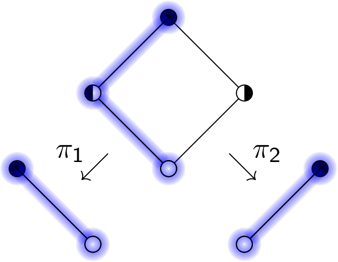

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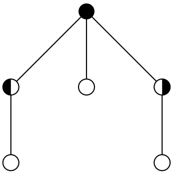

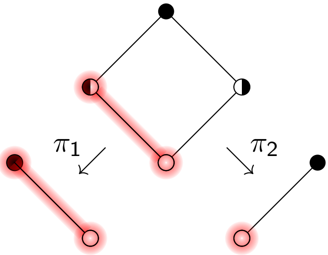

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Theorem (C. 2024)

- *If $\{X_i\}$ is a family of Esakia root systems, then $\bigotimes_i X_i$ is their product in the category of Esakia root systems.*
- *Let $\{G_i\}$ be a family of Gödel algebras and $\{X_i\}$ their dual Esakia root systems. Then $\bigoplus_i G_i$ is dual to $\bigotimes_i X_i$.*

Free algebras in varieties of Gödel algebras

The proper subvarieties of the variety of Gödel algebras form a countable chain $\mathsf{GA}_0 \subseteq \mathsf{GA}_1 \subseteq \cdots \subseteq \mathsf{GA}_n \subseteq \ldots$, where each GA_n consists of the Gödel algebras validating the bounded depth n axiom bd_n:

$$\mathsf{bd}_0 \coloneqq \bot \qquad \mathsf{bd}_{n+1} \coloneqq x_{n+1} \vee (x_{n+1} \to \mathsf{bd}_n)$$

Theorem (C. 2024)

The dual descriptions of free Gödel algebras and coproduct of Gödel algebras can be adapted to GA_n *by replacing* $\mathsf{CC}(X)$ *with its subspace*

$$\mathsf{CC}(X)_n = \{C \in \mathsf{CC}(X) \mid C \textit{ has size at most } n\}.$$

Interestingly, free algebras in GA_n are almost never bi-Heyting.

Corollary (C. 2024)

A free algebra in GA_n *is bi-Heyting iff it is finitely generated (iff it is finite).*

THANK YOU!