Free algebras and coproducts in varieties of Gödel algebras

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Free Heyting algebras

Intuitionistic logic is the logic of constructive mathematics and is obtained by weakening the principles of classical logic via the rejection of the law of excluded middle ($p \vee \neg p$).

Heyting algebras provide the algebraic semantics for the intuitionistic propositional calculus IPC.

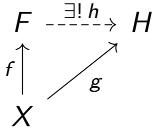
Definition

A Heyting algebra H is a distributive lattice equipped with a binary operation $\to$ satisfying

$$a \wedge b \leq c \quad \text{iff} \quad a \leq b \to c$$

for any $a, b, c \in H$.

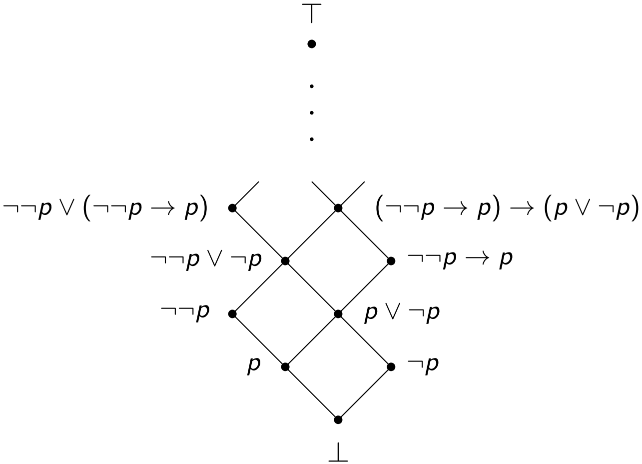
A Heyting algebra F is said to be free over a set X if there exists a map $f\colon X \to F$ such that for any Heyting algebra H and map $g\colon X \to H$, there is a unique Heyting homomorphism $h\colon F \to H$ with $g = h \circ f$.



Free Heyting algebras arise as Lindenbaum-Tarski algebras:

Let $\mathrm{Form}(X)$ be the set of formulas with variables from a set X and define $\varphi \sim \psi$ iff $\vdash_{\mathsf{IPC}} \varphi \leftrightarrow \psi$. Then $\mathrm{Form}(X)/{\sim}$ is a Heyting algebra free over X.

The free Heyting algebra over 1 generator is also known as the Rieger-Nishimura lattice.



The free Heyting algebra over 2 generators is very complicated.

Definition

An Esakia space is a Stone space X with a partial order $\leq$ such that:

- if $x \in X$, then $\uparrow x$ is closed;
- if $V \subseteq X$ is clopen, then $\downarrow V$ is clopen.

Esakia spaces		Heyting algebras
X	$\longrightarrow$	$\mathrm{ClopUp}(X)$
$\mathrm{Spec}(H)$	$\longleftarrow$	H

Theorem (Esakia duality 1974)

The category of Heyting algebras and Heyting homomorphisms is dually equivalent to the category of Esakia spaces and continuous p-morphisms.

The coloring technique was developed by Esakia and Grigolia in the 1970s to dually describe sets of generators.

Let $n \in \mathbb{N}$. The n-universal model X_n is built as follows:

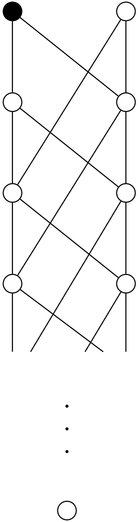
- each point of X_n is associated with a color, which is an element of $\mathcal{P}(\{1, \ldots, n\})$, and the coloring preserves the order;
- the layers of X_n are built each one at the time from the top;
- the top layer contains 2^n points, one for each color;
- two points of the same color cannot have the same elements as immediate successors;
- if a point has only one immediate successor, then its color should be strictly smaller than the one of its successor.

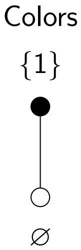
Theorem

- *F_n is isomorphic to a subalgebra of $\mathrm{Up}(X_n)$.*
- *X_n is the dense and open upset of the Esakia dual of F_n consisting of the points of finite depth (equivalently, $\uparrow x$ is finite).*

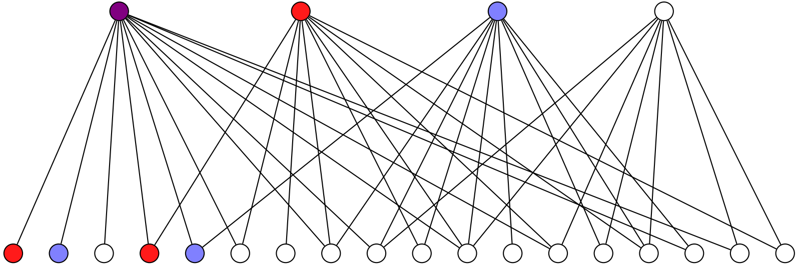
X_1 is also known as the Rieger-Nishimura ladder.

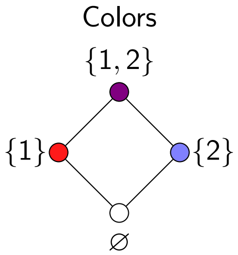
The Esakia dual of F_1 is the following.





However, X_2 is extremely complicated. Its third layer contains more than 250 000 points.





The Esakia dual of F_2 has the cardinality of the continuum.

Free Gödel algebras

The propositional Gödel-Dummett logic LC is obtained by adding the prelinearity axiom $(p \to q) \vee (q \to p)$ to IPC.

Gödel algebras provide the algebraic semantics for LC.

Definition

A Heyting algebra G is called a Gödel algebra if $(a \to b) \vee (b \to a) = 1$ for any $a, b \in G$.

We can think of LC as the extension of IPC in which "the truth values are linearly ordered". The variety of Gödel algebras is generated by the class of all finite Heyting chains or by any infinite Heyting chain.

Thus, LC is also the logic of $[0, 1]$ as a Heyting chain, and hence it can also be thought of as a fuzzy logic. In fact, LC is the t-norm fuzzy logic associated to the minimum t-norm.

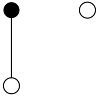
Free Gödel algebras arise as Lindenbaum-Tarski algebras for LC.

Proposition

An Esakia space X is dual to a Gödel algebra iff it is a root system; i.e., $\uparrow x$ is a chain for any $x \in X$. We call such spaces Esakia root systems.

The construction of the n-universal model can be adapted to LC by only adding points with a single immediate successor. So, the colors strictly decrease by moving down the layers.

The n-universal model for LC is finite and (once equipped with the discrete topology) coincides with the Esakia dual of the free Gödel algebra on n generators.



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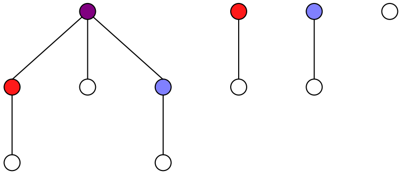

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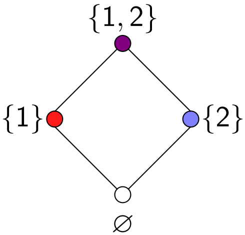
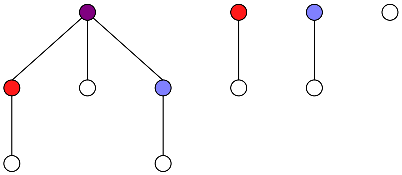


Proposition

The n-universal model for LC is isomorphic to the set of all nonempty chains in $\mathcal{P}(\{1, \ldots, n\})$ ordered by $C \leq D$ iff D is an upset of C.



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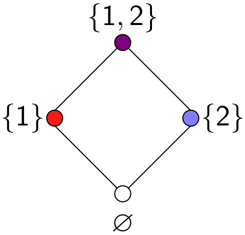

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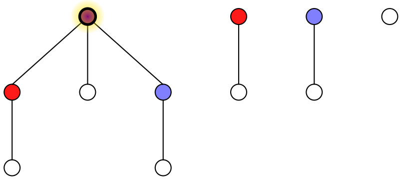

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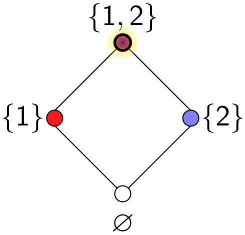

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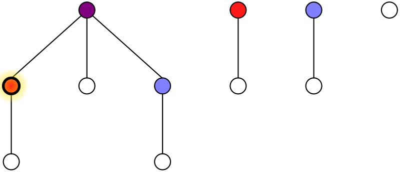

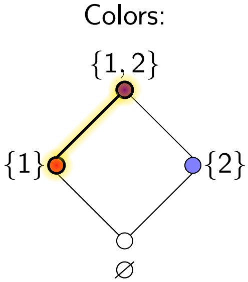

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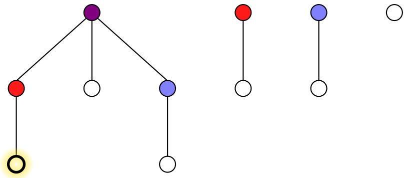

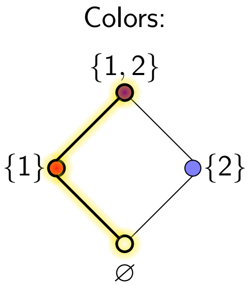

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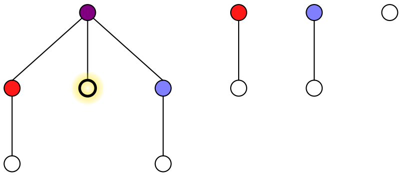

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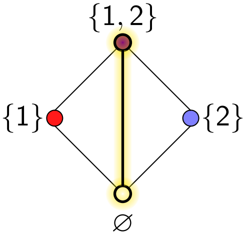

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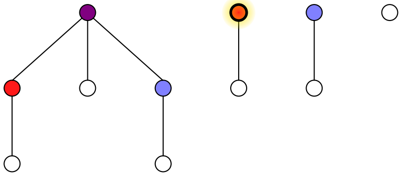

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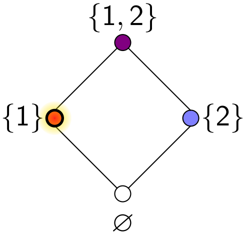

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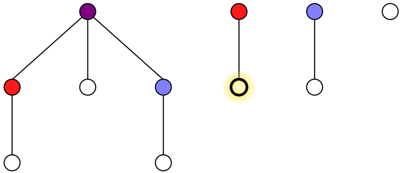

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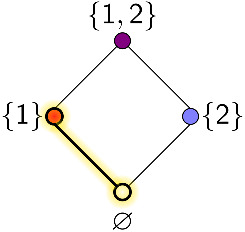

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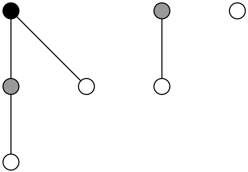
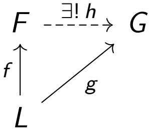


Which Gödel algebra is dual to what we get when we replace $\mathcal{P}(\{1, \ldots, n\})$ with an arbitrary finite poset P?

A Gödel algebra F is said to be free over a distributive lattice L if there exists a lattice homomorphism $f: L \to F$ such that for any Gödel algebra G and lattice homomorphism $g: L \to G$, there is a unique Heyting algebra homomorphism $h: F \to G$ with $g = h \circ f$.



Theorem (Aguzzoli, Gerla, and Marra 2008)

Let L be a finite distributive lattice and P a poset such that $L \cong \mathrm{Up}(P)$. The poset of all nonempty chains in P is the Esakia dual of the Gödel algebra free over L.

How to generalize this result to infinite distributive lattices?

Definition

A Priestley space is a Stone space X with a partial order $\leq$ such that

- $x \not\leq y$ implies there is U clopen upset such that $x \in U$ and $y \notin U$.

Priestley spaces		Distributive lattices
X	$\longrightarrow$	$\mathrm{ClopUp}(X)$
$\mathrm{Spec}(L)$	$\longleftarrow$	L

Theorem (Priestley duality 1972)

The category of distributive lattices and lattice homomorphisms is dually equivalent to the category of Priestley spaces and continuous order-preserving maps.

Definition

If X is a Priestley space, we define

$$\mathrm{CC}(X) := \{C \subseteq X \mid C \text{ is a nonempty closed chain}\}.$$

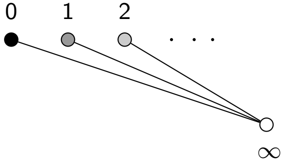
We order $\mathrm{CC}(X)$ by setting $C \leq D$ iff D is an upset of C.

We topologize $\mathrm{CC}(X)$ with the Vietoris topology, which is generated by the subbasis consisting of the sets $\Box V$ and $\Diamond V$ for any V clopen of X:

$$\Box V = \{C \in \mathrm{CC}(X) \mid C \subseteq V\},$$
$$\Diamond V = \{C \in \mathrm{CC}(X) \mid C \cap V \neq \varnothing\}.$$

Theorem (C. 2024)

- *If X is a Priestley space, then $\mathrm{CC}(X)$ is an Esakia root system.*
- *Let L be a distributive lattice and X its Priestley dual. Then the Gödel algebra free over L is dual to the Esakia space $\mathrm{CC}(X)$.*



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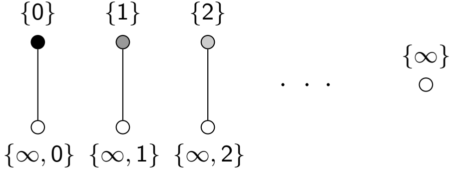
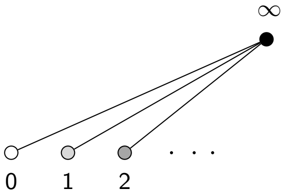


$\mathsf{CC}(X)$



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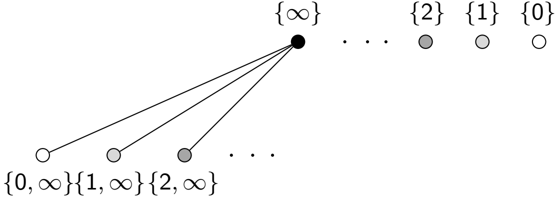


$\mathsf{CC}(X)$

Let $\mathbf{2}$ be the 2-element chain with the discrete topology. If S is a set, we consider $\mathbf{2}^S$ with the product topology and the product order.

Proposition

$\mathbf{2}^S$ is a Priestley space dual to the distributive lattice free over S.

Theorem (C. 2024)

The Gödel algebra free over a set S is dual to the Esakia space $\mathrm{CC}(\mathbf{2}^S)$.

Ghilardi in 1992 showed that Heyting algebras free over finitely many generators are bi-Heyting algebras.

Definition

Let L be a distributive lattice.

- L is a co-Heyting algebra if its order dual is a Heyting algebra.
- L is a bi-Heyting algebra if it is both a Heyting and a co-Heyting algebra.

Co-Heyting algebras are dual to co-Esakia spaces and bi-Heyting algebras are dual to bi-Esakia spaces.

Definition

Let X be a Priestley space.

- X is a co-Esakia space if $(X, \geq)$ is an Esakia space.
- X is a bi-Esakia space if it is both an Esakia and a co-Esakia space.

Free Gödel algebras over finitely many generators are finite, and so are bi-Heyting algebras.

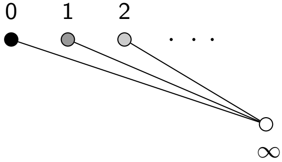
Theorem (C. 2024)

Let F be the Gödel algebra free over a distributive lattice L. Then F a bi-Heyting algebra iff L is a co-Heyting algebra.

Since free distributive lattices are co-Heyting algebras:

Corollary (C. 2024)

Free Gödel algebras are bi-Heyting algebras.



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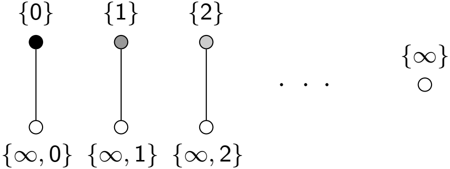

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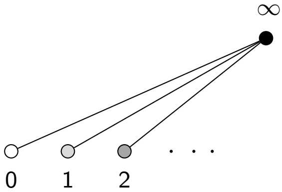

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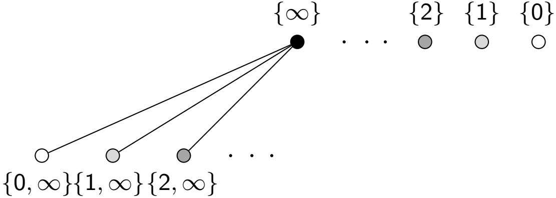
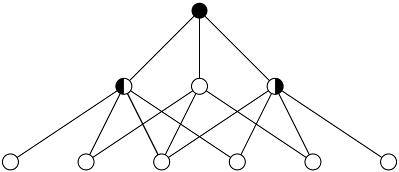


$\mathsf{CC}(X)$

Coproducts of Gödel algebras

Coproducts of Heyting algebras (dually, products of Esakia spaces) are complicated. In 2006 Grigolia gave a description of binary coproducts of finite Heyting algebras in dual terms.

If $\mathbf{2}$ is the 2-element chain, seen as an Esakia space, then $\mathbf{2} \times \mathbf{2}$ in the category of Esakia spaces is infinite. The following are its first 3 layers.

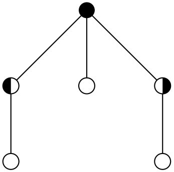


The size of the layers grow exponentially:

- the 4^{th} layer has 72 points
- and the 5^{th} layer has more than 10^{21} points.

Coproducts of Gödel algebras (dually, products of Esakia root systems) are much simpler. D'Antona and Marra in 2006 dually described the coproduct of two finite Gödel algebras, which is always finite.

The following is $\mathbf{2} \times \mathbf{2}$ in the category of Esakia root systems.

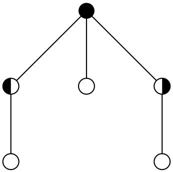


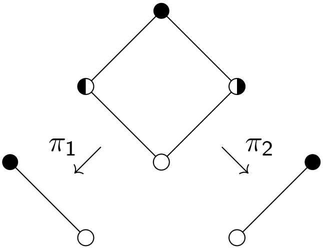
Let $\{X_i\}$ be a family of Esakia root systems. We denote by $\prod_i X_i$ their cartesian product with the product topology and the product order.

Definition

Let $\bigotimes_i X_i$ be the subspace of $\mathrm{CC}(\prod_i X_i)$ given by

$$\bigotimes_i X_i := \{C \in \mathrm{CC}(\prod_i X_i) \mid \pi_i[C] \text{ is a principal upset of } X_i \text{ for each } i \in I\}.$$





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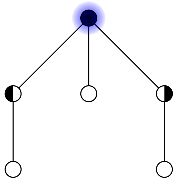

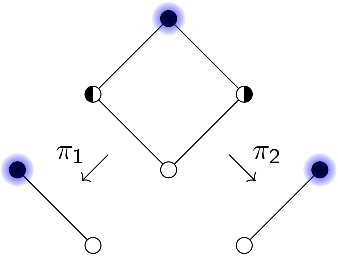

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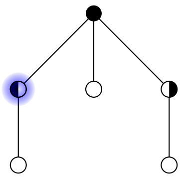

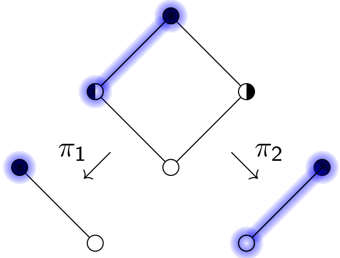

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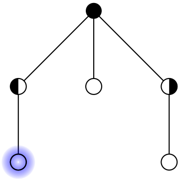

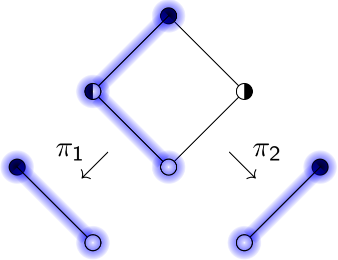

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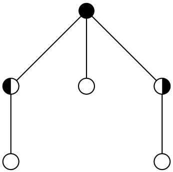
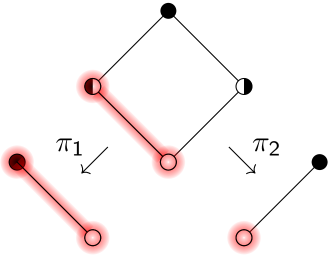
Let $\bigotimes_i X_i$ be the subspace of $\mathrm{CC}(\prod_i X_i)$ given by

$$\bigotimes_i X_i := \{C \in \mathrm{CC}(\prod_i X_i) \mid \pi_i[C] \text{ is a principal upset of } X_i \text{ for each } i \in I\}.$$





Theorem (C. 2024)

- *If $\{X_i\}$ is a family of Esakia root systems, then $\bigotimes_i X_i$ is their product in the category of Esakia root systems.*
- *Let $\{G_i\}$ be a family of Gödel algebras and $\{X_i\}$ their dual Esakia root systems. Then $\bigoplus_i G_i$ is dual to $\bigotimes_i X_i$.*

The depth (or height) of a poset is the sup of the lengths of its finite chains.

Theorem (C. 2024)

Let $\{X_i\}$ be a family of nonempty Esakia root systems. Then $\bigotimes_i X_i$ has depth

$$1 + \sum_{i \in I} (d_i - 1),$$

where $d_i \in \mathbb{N} \cup \{\infty\}$ is the depth of X_i.

Free algebras in varieties of Gödel algebras

Each proper extension of LC is of the form $\mathsf{LC}_n := \mathsf{LC} + \mathsf{bd}_n$, where bd_n is the bounded depth n axiom for $n \in \mathbb{N}$.

$$\mathsf{bd}_0 := \bot \qquad \mathsf{bd}_{n+1} := x_{n+1} \vee (x_{n+1} \to \mathsf{bd}_n)$$

The algebraic semantics for LC_n is given by GA_n; the class of Gödel algebras validating bd_n.

Theorem

Esakia duality restricts to a duality between the category of GA_n-algebras and the category of Esakia root systems of depth $\leq n$.

Theorem (C. 2024)

The dual descriptions of free Gödel algebras and coproduct of Gödel algebras can be adapted to GA_n *by replacing* $\mathsf{CC}(X)$ *with its subspace*

$$\mathsf{CC}(X)_n = \{C \in \mathsf{CC}(X) \mid C \text{ has size at most } n\}.$$

Interestingly, free GA_n-algebras are almost never bi-Heyting.

Corollary (C. 2024)

The GA_n*-algebra free over a set* S *is bi-Heyting iff* S *is finite.*

THANK YOU!