On the lack of colimits in various categories of BAOs and Heyting algebras

Luca Carai (University of Milan)

Joint work with Marco Abbadini and Guram Bezhanishvili

22 December 2025

6th ItaCa Workshop, University of Milan

The algebra of topology

A framework for an algebraic treatment of portions of topology.

Investigate properties of topological spaces by studying the algebraic structures induced on various collections of subsets of a topological space.

McKinsey and Tarski (1944) initiated the study of closure algebras as an abstraction of topological closure operators on powersets.

Closure algebras and topological spaces

Definition

- A Boolean algebra with an operator (BAO) is a Boolean algebra B together with a unary operator $\Diamond\colon B \to B$ preserving finite joins.
- A closure algebra is a BAO $(B, \Diamond)$ satisfying the Kuratowski's axioms:

$$x \leq \Diamond x \qquad\qquad \Diamond\Diamond x \leq \Diamond x$$

If X is a topological space, then $(\wp(X), \mathrm{cl})$ is a closure algebra.

Theorem (McKinsey-Tarski 1944)

An equation holds in $(\wp(X), \mathrm{cl})$ *for every topological space* X *iff it holds in every closure algebra.*

As a corollary, we obtain a topological semantics for the propositional modal logic S4.

Heyting algebras and topological spaces

Definition

A Heyting algebra is a bounded (distributive) lattice equipped with a binary operation $\to$ satisfying

$$x \leq y \to z \iff x \wedge y \leq z$$

If X is a topological space, then $\mathcal{O}(X)$ is a Heyting algebra with $U \to V = \text{int}((X \setminus U) \cup V)$.

Theorem (Stone and Tarski 1938)

An equation holds in $\mathcal{O}(X)$ for every topological space X iff it holds in every Heyting algebra.

As a corollary, we obtain a topological semantics for the intuitionistic propositional logic.

Frames and pointfree topology

Note that the Heyting algebra $\mathcal{O}(X)$ is always complete.

Definition

- A frame is a complete Heyting algebra or, equivalently, a complete lattice satisfying the join infinite distributive law

$$\left(\bigvee_i a_i\right) \wedge b = \bigvee_i (a_i \wedge b).$$

- Let **Frm** be the category of frames and bounded lattice homomorphisms that preserve arbitrary joins.

Let **Top** be the category of topological spaces and continuous maps.

Theorem

*$\mathcal{O}$: **Top** $\to$ **Frm** is a contravariant functor and is part of an adjunction which restricts to a dual equivalence between the full subcategories of sober topological spaces and of spatial frames.*

McKinsey-Tarski algebras

Note that also the closure algebra $(\wp(X), \mathrm{cl})$ is always complete.

Definition (Bezhanishvili-Raviprakash 2023)

- A McKinsey-Tarski algebra (MT-algebra) is a complete closure algebra.
- Let **MT** be the category of MT-algebras and complete Boolean homomorphisms that are stable; i.e., satisfy $\Diamond f(x) \leq f(\Diamond x)$.

As observed by Naturman in 1990, this framework is sufficient to capture all topological spaces.

Theorem

The full subcategory of **MT** *consisting of atomic MT-algebras is dually equivalent to* **Top**.

Theorem (Bezhanishvili-Raviprakash 2023)

Every frame can be realized as the sublattice of $\Box$-fixpoints of some MT-algebra, where $\Box = \neg\Diamond\neg$.

Frames and McKinsey-Tarski algebras

While frames provide a pointfree approach to study the open subsets of topological spaces, MT-algebras are a pointfree tool to investigate topological closure operators of topological spaces.

Theorem

Frm *is equationally presentable and the forgetful functor* **Frm** $\to$ **Set** *has a left adjoint. In particular,* **Frm** *is complete and cocomplete.*

Theorem (Melzer)

MT *is complete.*

GOAL: We show that **MT** and several related categories lack colimits.

McKinsey-Tarski algebras and complete stable morphisms

Theorem

The category **MT** *of MT-algebras lacks some countable copowers.*

Sketch of the proof.

- The category **CBA** of complete Boolean algebras and complete Boolean homomorphism does not have free objects over infinite sets.
- If A is the four-element Boolean algebra (free **CBA** over 1 generator), then infinite copowers of A do not exist in **CBA**.
- The forgetful functor **MT** $\to$ **CBA** is a left adjoint, and hence preserves colimits.
- A can be extended to an MT-algebra B.
- Coproducts of infinitely many copies of B don't exist in **MT**.

Theorem

The category $\mathbf{CBAO_{st}}$ *of complete Boolean algebras with operators and complete stable homomorphisms lacks some countable copowers.*

Similar results for several subcategories of $\mathbf{CBAO_{st}}$ hold.

McKinsey-Tarski algebras and stable morphisms

Theorem

The category $\mathbf{MT_{st}}$ *of MT-algebras and stable morphisms lacks some binary copowers.*

Sketch of the proof.

- Let **cBA** be the category of complete Boolean algebras and Boolean homomorphisms.
- The forgetful functor $\mathbf{MT_{st}} \to \mathbf{cBA}$ is a left adjoint and surjective.
- To show that **cBA** lacks binary copowers, we exploit the dual equivalence between **cBA** and the category **ED** of extrem**ally** disconnected Stone spaces and continuous maps.
- The category **ED** lacks some binary powers; e.g., $\beta\mathbb{N} \times \beta\mathbb{N}$.

Theorem

The category $\mathbf{cBAO_{st}}$ *of complete Boolean algebras with operators and stable homomorphisms lacks some binary copowers.*

Similar results for several subcategories of $\mathbf{cBAO_{st}}$ hold.

Closure algebras and stable morphisms

Theorem

The category $\mathbf{CA_{st}}$ *of closure algebras and stable morphisms lacks some coequalizers.*

Sketch of the proof.

- $\mathbf{CA_{st}}$ is dually equivalent to the category $\mathbf{StoneC_Q}$ of Stone spaces equipped with a continuous quasi order and order preserving continuous maps.
- Both forgetful functors of $\mathbf{StoneC_Q}$ into the categories of topological spaces and of quasi ordered sets preserve limits.
- We found two morphisms in $\mathbf{StoneC_Q}$ without an equalizer.

Theorem

The category $\mathbf{BAO_{st}}$ *of Boolean algebras with operators and stable homomorphisms lacks some coequalizers.*

Similar results for several subcategories of $\mathbf{BAO_{st}}$ hold.

Frames and lattice homomorphisms

Theorem

The category $\mathbf{Frm_{BL}}$ of frames and bounded lattice homomorphisms lacks some binary copowers.

Sketch of the proof.

- $\mathbf{Frm_{BL}}$ is dually equivalent to the category **Loc** of localic spaces and order preserving continuous maps.
- The embedding of the category **ED** of extremally disconnected spaces into **Loc** reflects products.
- We have seen that **ED** lacks some binary powers; e.g., $\beta\mathbb{N} \times \beta\mathbb{N}$.

Similar proof strategies yield the following results.

Theorem

- *The category $\mathbf{Frm_{HA}}$ of frames and Heyting homomorphisms lacks some binary copowers.*
- *The category $\mathbf{HA_{BL}}$ of Heyting algebras and bounded lattice homomorphisms lacks some coequalizers.*

THANK YOU!