# On the universal theory of the free pseudocomplemented distributive lattice

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# Intuitionistic logic

- Logic of constructive mathematics that has its origins in Brouwer's criticism of the use of the law of excluded middle ($p \vee \neg p$).
- It is obtained by weakening the principles of classical logic via the rejection of the law of excluded middle.
- Various semantic tools have been developed to study intuitionistic logic: algebraic, relational, and topological.

We denote by IPC the intuitionistic propositional calculus. Formulas in the language of IPC are built up from infinitely countably many propositional variables using $\wedge, \vee, \rightarrow, \bot, \top$. The negation $\neg$ is defined as an abbreviation $\neg\varphi := \varphi \rightarrow \bot$.

When a propositional formula $\varphi$ is intuitionistically valid we write $\vdash_{\mathsf{IPC}} \varphi$.

# Admissible rules

A multiconclusion rule is an expression of the form $\Gamma \Rightarrow \Delta$, where $\Gamma, \Delta$ are finite sets of formulas.

The expression $\Gamma \Rightarrow \Delta$ should be read as "if every formula in $\Gamma$ holds, then some formula in $\Delta$ holds".

When $\Delta = \{\delta\}$, then we write $\Gamma \Rightarrow \delta$ and call it a single-conclusion rule.

**Definition**

We say that a rule $\Gamma \Rightarrow \Delta$ is admissible in a logic L if for every substitution $\sigma$ we have that:

$$\vdash_{\mathsf{L}} \sigma(\gamma) \text{ for every } \gamma \in \Gamma, \text{ then there exists } \delta \in \Delta \text{ such that } \vdash_{\mathsf{L}} \sigma(\delta).$$

The Kreisel-Putnam rule

$$\neg p \to q \vee r \Rightarrow (\neg p \to q) \vee (\neg p \to r)$$

is admissible in IPC, although $\nvdash_{\mathsf{IPC}} (\neg p \to q \vee r) \to (\neg p \to q) \vee (\neg p \to r)$.

# Heying algebras

The variety HA of Heyting algebras provide the algebraic semantics for IPC.

## Definition

A Heyting algebra $H$ is a (bounded) distributive lattice equipped with a binary operation $\to$ satisfying

$$a \wedge b \leq c \quad \text{iff} \quad a \leq b \to c$$

for any $a, b, c \in H$.

Note that there is a correspondence between terms in the language of Heyting algebras and formulas in the language of IPC.

## Theorem (Algebraic completeness of IPC)

*Let $t_\varphi$ be a term corresponding to a formula $\varphi$. Then*

$$\mathrm{HA} \vDash t_\varphi = 1 \qquad \textit{iff} \qquad \vdash_{\mathrm{IPC}} \varphi.$$

# Free HA and admissible rules

Since Heyting algebras form a variety, for every cardinal $\kappa$ there exists the free Heyting algebra $\boldsymbol{F}_{\mathsf{HA}}(\kappa)$ over $\kappa$ generators.

$\boldsymbol{F}_{\mathsf{HA}}(\aleph_0)$ can be constructed by quotienting the set of all formulas by setting two formulas $\varphi$ and $\psi$ equivalent iff $\vdash_{\mathsf{IPC}} \varphi \leftrightarrow \psi$.

In particular, $\vdash_{\mathsf{IPC}} \varphi$ iff the equivalence class of $\varphi$ is the top of $\boldsymbol{F}_{\mathsf{HA}}(\aleph_0)$.

A substitution $\sigma$ can be thought as an infinite tuple $(\sigma(p_1), \sigma(p_2), \dots)$ of elements of $\boldsymbol{F}_{\mathsf{HA}}(\aleph_0)$. Therefore, if $\varphi$ is a formula that corresponds to a term $t_\varphi$, we have that $\sigma(\varphi)$ corresponds to the term $t_\varphi(\sigma(p_1), \sigma(p_2), \dots)$.

**Theorem**

*A rule $\Gamma \Rightarrow \Delta$ is admissible in* IPC *iff the universal first-order sentence*

$$\forall \overline{x} \left( (t_{\gamma_1} = 1 \ \& \ \cdots \ \& \ t_{\gamma_n} = 1) \Rightarrow (t_{\delta_1} = 1 \ or \ \cdots \ or \ t_{\delta_m} = 1) \right)$$

*holds in $\boldsymbol{F}_{\mathsf{HA}}(\aleph_0)$.*

# Free Heyting algebras and admissible rules

Every universal first-order sentence in the language of Heyting algebras is equivalent to a conjunction of sentences of the form

$$\forall \overline{x} \Big( (t_{\gamma_1} = 1 \,\&\, \cdots \,\&\, t_{\gamma_n} = 1) \Rightarrow (t_{\delta_1} = 1 \text{ or } \cdots \text{ or } t_{\delta_m} = 1) \Big).$$

Therefore, the universal theory of $\boldsymbol{F}_{\mathrm{HA}}(\aleph_0)$, i.e., the set $\mathrm{Th}_{\forall}(\boldsymbol{F}_{\mathrm{HA}}(\aleph_0))$ of universal first-order sentences that hold in $\boldsymbol{F}_{\mathrm{HA}}(\aleph_0)$, give all the information on admissible multiconclusion rules of IPC.

For example, the rule $p \vee q \Rightarrow \{p, q\}$ is admissible in IPC as it corresponds to the sentence $\forall x, y\, (x \vee y = 1 \Rightarrow (x = 1 \text{ or } y = 1))$, which holds in $\boldsymbol{F}_{\mathrm{HA}}(\aleph_0)$ (because free HA are finitely subdirectly irreducible).

Similarly, quasiequations that hold in $\boldsymbol{F}_{\mathrm{HA}}(\aleph_0)$ correspond to single-conclusion rules that are admissible in IPC.

# Decidability of admissibility and bases of admissible rules

**Theorem (Rybakov 1989, 1985)**

*The universal theory of $\boldsymbol{F}_{\mathrm{HA}}(\aleph_0)$ is decidable (its elementary theory is not).*

While the universal theory of $\boldsymbol{F}_{\mathrm{HA}}(\aleph_0)$ is not finitely axiomatizable, Jěrábek in 2008 provided an independent infinite axiomatization (i.e., a basis of admissible multiconclusion rules).

Iemhoff in 2001 provided an independent infinite axiomatization of the quasiequational theory of $\boldsymbol{F}_{\mathrm{HA}}(\aleph_0)$ (i.e., a basis of admissible single-conclusion rules) answering affirmatively a conjecture by de Jongh and Visser.

# $\mathsf{IPC}^-$ and Pseudocomplemented distributive lattices

Let $\mathsf{IPC}^-$ be the fragment of IPC consisting of the propositional intuitionistic validities containing only the connectives $\wedge, \vee, \neg, \bot, \top$.

## Definition

A pseudocomplemented distributive lattice $P$ is a distributive lattice equipped with a unary operation $\neg$ satisfying for any $a, b \in P$:

$$a \wedge b = 0 \quad \text{iff} \quad a \leq \neg b.$$

They are the $(\wedge, \vee, \neg, 0, 1)$-subreducts of Heyting algebras. The variety PDL provides an algebraic semantics for $\mathsf{IPC}^-$.

## Theorem (Algebraic completeness of $\mathsf{IPC}^-$)

*Let $t_\varphi$ be a term corresponding to a formula $\varphi$. Then*

$$\mathsf{PDL} \models t_\varphi = 1 \qquad \textit{iff} \qquad \vdash_{\mathsf{IPC}^-} \varphi.$$

# Our goals

Let $\boldsymbol{F}_{\mathrm{PDL}}(\aleph_0)$ be the free pseudocomplemented distributive lattice over $\aleph_0$ generators.

Our goals are:

- Determine whether the universal theory of $\boldsymbol{F}_{\mathrm{PDL}}(\aleph_0)$ is decidable.
- Provide an axiomatization of the universal theory of $\boldsymbol{F}_{\mathrm{PDL}}(\aleph_0)$.

# $\boldsymbol{F}_{\mathsf{PDL}}(\aleph_0)$ and admissible rules

IPC$^-$ is **not** algebraizable in the sense of Blok and Pigozzi.

The reason is essentially that you cannot always turn an equation $t = s$ in the language of PDL into the validity of a formula in IPC$^-$ because of the lack of the implication connective.

Rules for IPC$^-$ correspond to universal first-order sentences of the form

$$\forall \overline{x} \left( (t_{\gamma_1} = 1 \ \& \ \cdots \ \& \ t_{\gamma_n} = 1) \Rightarrow (t_{\delta_1} = 1 \text{ or } \cdots \text{ or } t_{\delta_m} = 1) \right),$$

while generic universal first-order sentences are conjunctions of

$$\forall \overline{x} \left( (t_1 = t_1' \ \& \ \cdots \ \& \ t_n = t_n') \Rightarrow (s_1 = s_1' \text{ or } \cdots \text{ or } s_m = s_m') \right),$$

which are more general.

The decidability of $\mathrm{Th}_\forall(\boldsymbol{F}_{\mathsf{PDL}}(\aleph_0))$ yields the decidability of admissibility in IPC$^-$. However, the axiomatization doesn't have to consist of universal sentences of the first kind (it won't), and so it doesn't correspond to a basis of admissible rules.

# Strategy

- Use a duality for finite pseudocomplemented distributive lattices to describe the finite members of PDL that embed into $\boldsymbol{F}_{\mathrm{PDL}}(\aleph_0)$.
- Exploit the local finiteness of PDL to obtain a description of the models of $\mathrm{Th}_{\forall}(\boldsymbol{F}_{\mathrm{PDL}}(\aleph_0))$; i.e., the members of the universal class $\mathbb{U}(\boldsymbol{F}_{\mathrm{PDL}}(\aleph_0))$ generated by $\boldsymbol{F}_{\mathrm{PDL}}(\aleph_0)$.
- Use the description of the members of $\mathbb{U}(\boldsymbol{F}_{\mathrm{PDL}}(\aleph_0))$ to derive the decidability and the axiomatization.

# Duality for finite PDL

A map $p\colon X \to Y$ between finite posets is said to be a weak p-morphism when it is order preserving and for all $x \in X$ and $y \in \max Y$,

$$\text{if } p(x) \leq y \text{, there exists } z \in \max{\uparrow}x \text{ such that } p(z) = y.$$

As a consequence of a duality for PDL due to Priestley (1975) we obtain.

**Theorem**

*The category of finite pseudocomplemented distributive lattices is dually equivalent to the category of finite posets and weak p-morphisms.*

| finite posets | | finite PDL |
|---|---|---|
| $X$ | $\longrightarrow$ | $(\mathrm{Up}(X), \subseteq)$ |
| $(\mathrm{Jirr}(A), \geq)$ | $\longleftarrow$ | $A$ |

# Posets with free skeleton

**Theorem (C. & Moraschini 2025)**

*Let $A$ be a finite PDL. Then $A$ embeds into $\boldsymbol{F}_{\mathrm{PDL}}(\aleph_0)$ if and only if its dual poset has a free skeleton.*

A poset $X$ with minimum $\bot$ is said to have a free skeleton when the following hold:

- for all $x \in X$ and nonempty $Y \subseteq \max{\uparrow}x$ there exists an element $s_{x,Y} \in {\uparrow}x$ such that
$$Y = \max{\uparrow}s_{x,Y};$$
- for all $x \in X$ and nonempty $Y, Z \subseteq \max{\uparrow}x$,
$$Y \subseteq Z \text{ implies } s_{x,Z} \leq s_{x,Y};$$
- for all $x \in X$ and nonempty $Y \subseteq \max X$,
$$\max{\uparrow}x \subseteq Y \text{ implies } s_{\bot,Y} \leq x.$$

# Duals of free finitely generated PDL

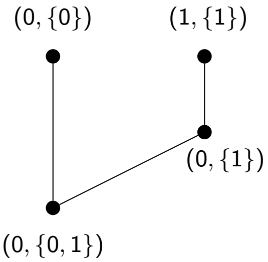
**Theorem (Urquhart 1973 (see also Davey & Goldberg 1980))**

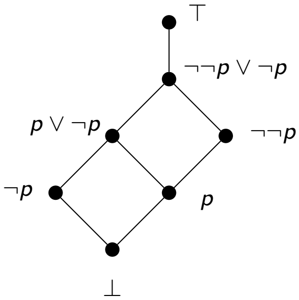
*The dual of the free n-generated pseudocomplemented distributive lattice* $\boldsymbol{F}_{\mathrm{PDL}}(n)$ *is the poset* $\mathrm{P}(n)$ *with universe*

$$\{(x, C) \in 2^n \times \wp(2^n) : \varnothing \neq C \subseteq {\uparrow}x\},$$

*ordered as follows:*

$$(x, C) \leq (y, D) \iff x \leq y \text{ and } C \supseteq D.$$



# Posets with free skeleton and free PDL

**Theorem (C. & Moraschini 2025)**

*Let $A$ be a finite PDL. Then $A$ embeds into $\boldsymbol{F}_{\mathrm{PDL}}(\aleph_0)$ if and only if its dual poset has a free skeleton.*

*Sketch of the proof:* Let $A \in \mathrm{PDL}$ be finite.

- $A$ embeds into $\boldsymbol{F}_{\mathrm{PDL}}(\aleph_0)$ iff it embeds into $\boldsymbol{F}_{\mathrm{PDL}}(n)$ for some $n$.
- $A$ embeds into $\boldsymbol{F}_{\mathrm{PDL}}(n)$ iff its dual is a weak p-morphic image of $\mathrm{P}(n)$.
- $\mathrm{P}(n)$ has a free skeleton: for all $(x, C) \in \mathrm{P}(n)$ and nonempty $Y \subseteq \max{\uparrow}(x, C)$ take $s_{x,Y} = (x, D)$, where $Y = \{(d, \{d\}) : d \in D\}$.
- Onto weak p-morphisms $\mathrm{P}(n) \to X$ transport the free skeleton structure of $\mathrm{P}(n)$ to $X$.
- If a finite poset $X$ has a free skeleton then you can build an onto weak p-morphism $\mathrm{P}(n) \to X$ for some $n$ (this is the hard part).

# A useful universal algebraic fact

Recall that for a class of algebras $\mathsf{K}$ the class of models of $\mathrm{Th}_\forall(\mathsf{K})$ is $\mathbb{U}(\mathsf{K}) = \mathbb{I}\mathbb{S}\mathbb{P}_{\mathsf{u}}(\mathsf{K})$.

**Theorem**

*Let* $\mathsf{V}$ *be a locally finite variety and* $\mathsf{K} \subseteq \mathsf{V}$. *Then*

$$\mathbb{U}(\mathsf{K}) = \{A \in \mathsf{V} : B \in \mathbb{I}\mathbb{S}(\mathsf{K}) \text{ for every finite subalgebra } B \text{ of } A\}.$$

Main ingredient for the proof: each algebra $A$ embeds into an ultraproduct of its finitely generated subalgebras.

# Models of $\mathrm{Th}_\forall(\boldsymbol{F}_{\mathrm{PDL}}(\aleph_0))$

Recall: the class of models of $\mathrm{Th}_\forall(\boldsymbol{F}_{\mathrm{PDL}}(\aleph_0))$ is $\mathbb{U}(\boldsymbol{F}_{\mathrm{PDL}}(\aleph_0))$.

**Theorem (C. & Moraschini 2025)**

$\mathbb{U}(\boldsymbol{F}_{\mathrm{PDL}}(\aleph_0)) =$
$\{A \in \mathrm{PDL} : \textit{duals of all finite subalgebras of } A \textit{ have a free skeleton}\}$.

It is well known that for all varieties V and infinite cardinal $\kappa$ we have

$$\mathbb{U}(\boldsymbol{F}_{\mathrm{V}}(\aleph_0)) = \mathbb{U}(\boldsymbol{F}_{\mathrm{V}}(\kappa)) = \mathbb{U}(\{\boldsymbol{F}_{\mathrm{V}}(n) : n \in \mathbb{Z}^+\}),$$

or equivalently

$$\mathrm{Th}_\forall(\boldsymbol{F}_{\mathrm{V}}(\aleph_0)) = \mathrm{Th}_\forall(\boldsymbol{F}_{\mathrm{V}}(\kappa)) = \mathrm{Th}_\forall(\{\boldsymbol{F}_{\mathrm{V}}(n) : n \in \mathbb{Z}^+\}).$$

Therefore, we also obtain a characterization of the members of $\mathbb{U}(\boldsymbol{F}_{\mathrm{PDL}}(\kappa))$ for every infinite cardinal $\kappa$ and of $\mathbb{U}(\{\boldsymbol{F}_{\mathrm{PDL}}(n) : n \in \mathbb{Z}^+\})$.

# Axiomatization of $\mathrm{Th}_\forall(\boldsymbol{F}_{\mathrm{PDL}}(\aleph_0))$

The atomic diagram of a finite pseudocomplemented distributive lattice $A = \{a_1, \ldots, a_n\}$ is the set of equations in the variables $x_1, \ldots, x_n$

$$\{f(x_{i_1}, \ldots, x_{i_m}) \approx x_k : f \in \{\wedge, \vee, \neg, 0, 1\} \text{ and } f^A(a_{i_1}, \ldots, a_{i_m}) = a_k\};$$

together with the negated equations

$$\{x_m \not\approx x_k : m < k \leq n\}.$$

**Theorem (C. & Moraschini 2025)**

*The theory* $\mathrm{Th}_\forall(\boldsymbol{F}_{\mathrm{PDL}}(\aleph_0))$ *is recursively axiomatizable by*

$$\Sigma \cup \{\neg\exists x_1, \ldots, x_n \bigsqcap \mathrm{diag}(A) : A \in \mathrm{PDL} \textit{ is finite and its dual lacks a free skeleton}\},$$

*where* $\Sigma$ *is a finite set of axioms of* PDL.

We have also obtained an alternative axiomatization that, although still infinite, captures the idea of having a free skeleton in a more concrete way.

# Decidability

**Theorem (C. & Moraschini 2025)**

$\mathrm{Th}_\forall(\boldsymbol{F}_{\mathrm{PDL}}(\aleph_0))$ *is decidable.*

*Sketch of the proof:*

- We have obtained a recursive axiomatization of $\mathrm{Th}_\forall(\boldsymbol{F}_{\mathrm{PDL}}(\aleph_0))$.
- Let V be a finitely axiomatizable and locally finite variety of finite type. If $\mathrm{Th}_\forall(\boldsymbol{F}_{\mathrm{V}}(\aleph_0))$ is recursively axiomatizable, then it is also decidable.
- We conclude that $\mathrm{Th}_\forall(\boldsymbol{F}_{\mathrm{PDL}}(\aleph_0))$ is decidable.

**Corollary**

*Admissibility of multiconclusion rules in* $\mathrm{IPC}^-$ *is decidable.*

**Corollary**

$\mathrm{Th}_\forall(\boldsymbol{F}_{\mathrm{PDL}}(\kappa))$ *for* $\kappa$ *infinite, and* $\mathrm{Th}_\forall(\{\boldsymbol{F}_{\mathrm{PDL}}(n) : n \in \mathbb{Z}^+\})$ *are decidable.*

Idziak in 1987 showed that the elementary theory of $\{\boldsymbol{F}_{\mathrm{PDL}}(n) : n \in \mathbb{Z}^+\}$ is undecidable.

# Derivable and admissible rules in IPC

IPC can also be defined as a consequence relation. When there is an intuitionistically valid proof of formula $\delta$ from a set of formulas $\Gamma$, we write $\Gamma \vdash_{\mathsf{IPC}} \delta$.

A single-conclusion rule $\Gamma \Rightarrow \delta$ is called derivable in IPC if $\Gamma \vdash_{\mathsf{IPC}} \delta$.

The deduction theorem yields that $\Gamma \Rightarrow \delta$ is derivable in IPC iff $\vdash_{\mathsf{IPC}} \gamma_1 \wedge \cdots \wedge \gamma_n \to \delta$.

A derivable rule is always admissible, but the converse is not true in general. When that happens, the logic is said to be structurally complete.

The Kreisel-Putnam rule

$$\neg p \to q \vee r \Rightarrow (\neg p \to q) \vee (\neg p \to r)$$

is admissible, but not derivable, in IPC. So,

**Theorem**

IPC *is **not** structurally complete.*
*Algebraically:* $\mathsf{HA} \neq \mathbb{Q}(\boldsymbol{F}_{\mathsf{HA}}(\aleph_0))$.

# Derivable and admissible rules in $\mathsf{IPC}^-$

$\mathsf{IPC}^-$ can be defined also as a consequence relation as a fragment of IPC. For a set of formulas $\Gamma \cup \{\delta\}$ in the language of $\mathsf{IPC}^-$ we define $\Gamma \vdash_{\mathsf{IPC}^-} \delta$ iff $\Gamma \vdash_{\mathsf{IPC}} \delta$.

Recall that a rule $\Gamma \Rightarrow \Delta$ is admissible in $\mathsf{IPC}^-$ if for every substitution $\sigma$ we have that:

$\vdash_{\mathsf{IPC}^-} \sigma(\gamma)$ for every $\gamma \in \Gamma$, then there exists $\delta \in \Delta$ such that $\vdash_{\mathsf{IPC}^-} \sigma(\delta)$.

The substitution $\sigma$ ranges over the formulas in the language of $\mathsf{IPC}^-$. So, if $\Gamma \Rightarrow \Delta$ is admissible in $\mathsf{IPC}^-$, then it is not immediate that it is also admissible in IPC. Mints showed that in fact it is even derivable in IPC.

**Theorem (Mints 1976)**

$\mathsf{IPC}^-$ *is structurally complete.*
*Algebraically: for* $\Phi$ *"special quasiequation",* $\mathsf{PDL} \models \Phi$ *iff* $\boldsymbol{F}_{\mathsf{PDL}}(\aleph_0) \models \Phi$*.*

Nonetheless, $\mathsf{PDL} \neq \mathbb{Q}(\boldsymbol{F}_{\mathsf{PDL}}(\aleph_0))$ (shown by looking at the SI members).

MOLTES GRÀCIES!

MUCHAS GRACIAS!