A calculus for modal compact Hausdorff spaces

Luca Carai
University of Milan

joint work with N. Bezhanishvili, S. Ghilardi, and Z. Zhao

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Let X be a topological space, $x \in X$, and $v\colon \mathrm{Prop} \to \wp(X)$.

$x \vDash_v p$	iff	$x \in v(p)$
$x \vDash_v \neg\varphi$	iff	$x \nvDash_v \varphi$
$x \vDash_v \varphi \wedge \psi$	iff	$x \vDash_v \varphi$ and $x \vDash_v \psi$
$x \vDash_v \varphi \vee \psi$	iff	$x \vDash_v \varphi$ or $x \vDash_v \psi$
$x \vDash_v \Box\varphi$	iff	$x \in \mathrm{int}\{y \in X : y \vDash_v \varphi\}$

We write $X \vDash \varphi$, when $x \vDash_v \varphi$ for every $x \in X$ and $v\colon \mathrm{Prop} \to \wp(X)$.

Theorem (McKinsey-Tarski 1938, 1944)

$\mathrm{S4} \vdash \varphi$ *iff* $X \vDash \varphi$ *for every topological space* X.

Let X be a topological space, $x \in X$, and $v\colon \mathrm{Prop} \to \mathcal{O}(X)$.

$x \vDash_v p$	iff	$x \in v(p)$
$x \vDash_v \neg\varphi$	iff	$x \in \mathrm{int}\{y \in X : y \nvDash_v \varphi\}$
$x \vDash_v \varphi \wedge \psi$	iff	$x \vDash_v \varphi$ and $x \vDash_v \psi$
$x \vDash_v \varphi \vee \psi$	iff	$x \vDash_v \varphi$ or $x \vDash_v \psi$
$x \vDash_v \varphi \to \psi$	iff	$x \in \mathrm{int}\{y \in X : y \vDash_v \varphi \Rightarrow y \vDash_v \psi\}$

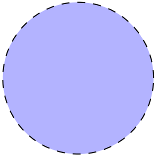
We write $X \vDash \varphi$, when $x \vDash_v \varphi$ for every $x \in X$ and $v\colon \mathrm{Prop} \to \mathcal{O}(X)$.

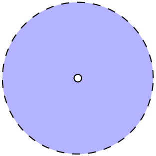
Theorem

$\mathrm{IPC} \vdash \varphi$ *iff* $X \vDash \varphi$ *for every topological space* X.

Definition

An open set U is called regular open if $U = \mathrm{int}(\mathrm{cl}(U))$.

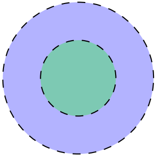




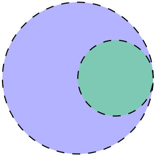
Let X be a topological space. Then the poset of its regular opens $\mathcal{RO}(X)$ ordered by inclusion is a complete Boolean algebra.

Definition (Well-inside relation)

Let $U, V \in \mathcal{RO}(X)$. We write $U \prec V$ iff $\mathrm{cl}(U) \subseteq V$.



$U \prec V$



$U \not\prec V$

The symmetric strict implication calculus S^2IC extends the classical propositional calculus with a binary connective $\rightsquigarrow$ of strict implication subject to some axioms and rules.

Let X be a topological space, $x \in X$, and $v \colon \mathrm{Prop} \to \mathcal{RO}(X)$.

$x \vDash_v p$	iff	$x \in v(p)$
$x \vDash_v \neg\varphi$	iff	$x \in \mathrm{int}\{y \in X : y \nvDash_v \varphi\}$
$x \vDash_v \varphi \wedge \psi$	iff	$x \vDash_v \varphi$ and $x \vDash_v \psi$
$x \vDash_v \varphi \vee \psi$	iff	$x \in \mathrm{int}(\mathrm{cl}\{y \in X : y \vDash_v \varphi \text{ or } y \vDash_v \psi\})$
$x \vDash_v \varphi \rightsquigarrow \psi$	iff	$\{y \in X : y \vDash_v \varphi\} \prec \{y \in X : y \vDash_v \psi\}$

Theorem (G. Bezhanishvili, N. Bezhanishvili, Santoli, Venema 2019)

$S^2IC \vdash \varphi$ *iff* $X \vDash \varphi$ *for every compact Hausdorff space* X.

They proved a stronger result: S^2IC is strongly complete with respect to compact Hausdorff spaces.

Modal compact Hausdorff spaces were introduced by G. Bezhanishvili, N. Bezhanishvili, and Harding in 2015 as a generalization of descriptive frames.

Definition

A binary relation R on a compact Hausdorff space X is said to be continuous provided

- $R[x]$ is closed for each $x \in X$;
- $R^{-1}[F]$ is closed for each closed $F \subseteq X$;
- $R^{-1}[U]$ is open for each open $U \subseteq X$.

We call a pair (X, R) a modal compact Hausdorff space if X is a compact Hausdorff space and R a continuous relation on X.

Definition

Let MS^2IC be the propositional modal system obtained by extending S^2IC with the axioms schemes:

- $\Box(\phi \to \psi) \to (\Box\phi \to \Box\psi)$;
- $(\phi \rightsquigarrow \psi) \to (\Box\phi \rightsquigarrow \Box\psi)$;

and the inference rule:

- $\dfrac{\phi}{\Box\phi}$.

Extending the semantic for S^2IC to modal compact Hausdorff spaces as follows:

$$x \vDash_v \Box\varphi \qquad \text{iff} \qquad x \in \mathrm{int}(\mathrm{cl}\{y \in X : yRz \Rightarrow z \vDash_v \varphi\})$$

Theorem (N. Bezhanishvili, C., Ghilardi, Zhao 2024)

$MS^2IC \vdash \varphi$ *iff* $X \vDash \varphi$ *for every modal compact Hausdorff space* X.

We proved a stronger result: MS^2IC is strongly complete with respect to modal compact Hausdorff spaces.

Definition

Let X a set, T a ternary relation on X and S a binary relation on X. We call (X, T, S) an $\mathrm{MS^2IC}$-frame if for every $x, y, z \in X$:

- $Txxx$;
- $Txyz \Rightarrow Txzy$;
- E_T given by xE_Ty iff $\exists z\,(Txyz)$ is an equivalence relation;
- $Txyz \;\&\; xE_Tw \Rightarrow Twyz$;
- $Txyz \;\&\; ySw \Rightarrow \exists u \in X : Txwu \;\&\; zSu$.

Let $v\colon \mathrm{Prop} \to \wp(X)$ be a valuation. We set

$$x \vDash_v \Box\varphi \quad \text{iff} \quad \forall y \in X\,(xSy \Rightarrow y \vDash_v \varphi)$$

$$x \vDash_v \varphi \rightsquigarrow \psi \quad \text{iff} \quad \forall y, z \in X\,(Txyz \;\&\; y \vDash_v \varphi \Rightarrow z \vDash_v \psi)$$

Theorem (N. Bezhanishvili, C., Ghilardi, Zhao 2024)

$\mathrm{MS^2IC} \vdash \varphi$ *iff it is valid in all* $\mathrm{MS^2IC}$*-frames.*

If X is a compact Hausdorff space, then $(\mathcal{RO}(X), \prec)$ is a de Vries algebra. In fact, every de Vries algebra arises this way up to isomorphism.

Theorem (de Vries duality, de Vries 1962)

The category of compact Hausdorff spaces and continuous functions is dually equivalent to the category of de Vries algebras.

If (X, R) is a modal compact Hausdorff space, then $(\mathcal{RO}(X), \prec, \Box)$ is a lower continuous modal de Vries algebra, where

$$\Box U = \mathrm{int}(\mathrm{cl}\{x \in X : R[x] \subseteq U\}).$$

Theorem (G. Bezhanishvili, N. Bezhanishvili, Harding 2015)

The category of modal compact Hausdorff spaces is dually equivalent to the category of lower continuous modal de Vries algebras.

Strategy of our completeness proof:

- Using Kripke semantics, we show that some nonstandard inference rules (Π_2-rules) are admissible in MS^2IC.
- We use these rules to prove that MS^2IC is strongly complete with respect to lower continuous modal de Vries algebras
- We then use the duality between modal compact Hausdorff spaces and lower continuous modal de Vries algebras to obtain strong completeness with respect to the class of modal compact Hausdorff spaces.

We also obtain that MS^2IC is strongly complete with respect to the classes of:

- lower continuous modal contact algebras;
- lower continuous modal compingent algebras;
- modal compact Hausdorff spaces that are zero-dimensional (i.e., descriptive frames).

THANK YOU!

The *symmetric strict implication calculus* S^2IC is the deductive system containing all the substitution instances of the theorems of the classical propositional calculus and of the axioms:

(A1) $(\bot \rightsquigarrow \phi) \wedge (\phi \rightsquigarrow \top)$;

(A2) $[(\phi \vee \psi) \rightsquigarrow \chi] \leftrightarrow [(\phi \rightsquigarrow \chi) \wedge (\psi \rightsquigarrow \chi)]$;

(A3) $[\phi \rightsquigarrow (\psi \wedge \chi)] \leftrightarrow [(\phi \rightsquigarrow \psi) \wedge (\phi \rightsquigarrow \chi)]$;

(A4) $(\phi \rightsquigarrow \psi) \to (\phi \to \psi)$;

(A5) $(\phi \rightsquigarrow \psi) \leftrightarrow (\neq \psi \rightsquigarrow \neq \phi)$;

(A8) $[\forall]\phi \to [\forall][\forall]\phi$;

(A9) $\neq [\forall]\phi \to [\forall]\neg[\forall]\phi$;

(A10) $(\phi \rightsquigarrow \psi) \leftrightarrow [\forall](\phi \rightsquigarrow \psi)$;

(A11) $[\forall]\phi \to (\neg[\forall]\phi \rightsquigarrow \bot)$;

and is closed under the inference rules

(MP) $\dfrac{\phi \quad \phi \to \psi}{\psi}$;

(R) $\dfrac{\phi}{[\forall]\phi}$.

A *contact algebra* is a pair $\mathbf{B} = (B, \prec)$, where B is a Boolean algebra and $\prec$ is a binary relation on B satisfying the following conditions:

(S1) $0 \prec 0$ and $1 \prec 1$;

(S2) $a \prec b$ and $a \prec c$ implies $a \prec b \wedge c$;

(S3) $a \prec c$ and $b \prec c$ implies $a \vee b \prec c$;

(S4) $a \leq b \prec c \leq d$ implies $a \prec d$;

(S5) $a \prec b$ implies $a \leq b$;

(S6) $a \prec b$ implies $\neg b \prec \neg a$.

A contact algebra $\mathbf{B} = (B, \prec)$ is called a *compingent algebra* if it satisfies the following two additional properties:

(S7) $a \prec b$ implies there is c with $a \prec c \prec b$;

(S8) $a \neq 0$ implies there is $b \neq 0$ with $b \prec a$.

A compingent algebra $\mathbf{B}$ is called a *de Vries algebra* if B is a complete Boolean algebra.

Let $(B, \prec)$ be a modal contact algebra and $\Box \colon B \to B$.
We call $\Box$ *de Vries multiplicative* if

- $\Box 1 = 1$;
- $a_1 \prec b_1$ and $a_2 \prec b_2$ implies that $\Box a_1 \wedge \Box a_2 \prec \Box(b_1 \wedge b_2)$.

We call $\Box$ *lower continuous* if $\bigvee\{\Box b : b \prec a\}$ exists and is equal to $\Box a$ for each $a \in B$.