La dualità di Baker-Beynon oltre la semisemplicità

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Baker-Beynon duality

Definition

A Riesz space V is an $\mathbb{R}$-vector space equipped with a lattice structure such that for every $u, v, w \in V$ and $0 \leq r \in \mathbb{R}$:

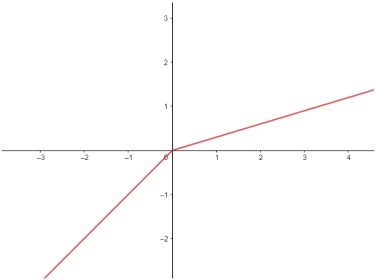
- if $u \leq v$, then $u + w \leq v + w$ and $ru \leq rv$.

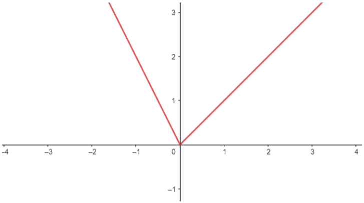
A map between Riesz spaces is a Riesz space homomorphism if it is a linear map and a lattice homomorphism.

Examples of Riesz spaces

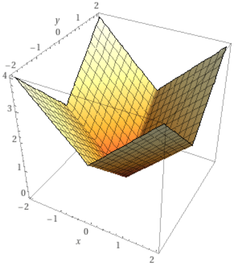
- $\mathbb{R}$
- $\mathbb{R}^X$ for a set X
- $\mathbb{R} \overrightarrow{\times} \mathbb{R}$ (lexicographic product)
- $C(X, \mathbb{R})$ for a topological space X

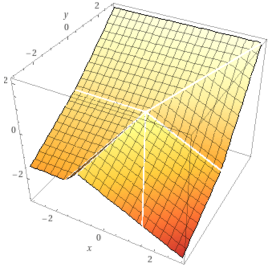
A continuous function $f\colon \mathbb{R}^\kappa \to \mathbb{R}$ is piecewise linear (homogeneous) if there exist $g_1, \ldots, g_n\colon \mathbb{R}^\kappa \to \mathbb{R}$ linear homogeneous functions (each in finitely many variables) such that for each $x \in \mathbb{R}^\kappa$ we have $f(x) = g_i(x)$ for some $i = 1, \ldots, n$.





A continuous function $f\colon \mathbb{R}^\kappa \to \mathbb{R}$ is piecewise linear (homogeneous) if there exist $g_1, \ldots, g_n\colon \mathbb{R}^\kappa \to \mathbb{R}$ linear homogeneous functions (each in finitely many variables) such that for each $x \in \mathbb{R}^\kappa$ we have $f(x) = g_i(x)$ for some $i = 1, \ldots, n$.





The piecewise linear functions $f\colon \mathbb{R}^\kappa \to \mathbb{R}$ form a Riesz space that we denote by $\mathrm{PWL}(\mathbb{R}^\kappa)$.

Theorem (Baker 1968)

Let κ be a cardinal. The free Riesz space on κ generators is isomorphic to $\mathrm{PWL}(\mathbb{R}^\kappa)$. *The free generators correspond to the projection maps onto each coordinate.*

If $X \subseteq \mathbb{R}^\kappa$, we let $\mathrm{PWL}(X) \coloneqq \{f|_X \text{ with } f \in \mathrm{PWL}(\mathbb{R}^\kappa)\}$.

Which Riesz spaces are isomorphic to $\mathrm{PWL}(X)$ for some $X \subseteq \mathbb{R}^\kappa$?

Definition

An ℓ-ideal in a Riesz space is a subgroup (linear subspace) I that is convex, i.e. $|a| \leq |b|$ and $b \in I$ imply $a \in I$.

Definition

- A nontrivial Riesz space A is simple if $\{0\}$ and A are the only ℓ-ideals of A.
- A Riesz space is semisimple if the intersection of all its maximal ℓ-ideals is $\{0\}$.

Proposition

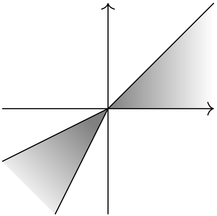
- *A Riesz space is simple iff it is isomorphic to $\mathbb{R}$.*
- *A Riesz space is semisimple iff it can be (subdirectly) embedded into a power of $\mathbb{R}$.*
- PWL(X) *is semisimple for any $X \subseteq \mathbb{R}^{\kappa}$.*

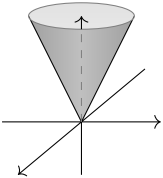
Theorem (Baker 1968)

Every semisimple Riesz space is isomorphic to PWL(C) *for some closed cone* $C \subseteq \mathbb{R}^\kappa$.

Definition

A nonempty subset $C \subseteq \mathbb{R}^\kappa$ is a closed cone if it is closed under multiplication by nonnegative scalars and it is topologically closed in $\mathbb{R}^\kappa$ with the euclidean topology.





This representation result extends to Baker-Beynon duality.

Let $\mathscr{F}_\kappa$ be the free Riesz space over κ generators.
For any $T \subseteq \mathscr{F}_\kappa$ and $S \subseteq \mathbb{R}^\kappa$, we define the following operators.

$$\mathrm{V}(T) = \{x \in \mathbb{R}^\kappa \mid t(x) = 0 \text{ for all } [t] \in T\}$$
$$\mathrm{I}(S) = \{[t] \in \mathscr{F}_\kappa \mid t(x) = 0 \text{ for all } x \in S\}.$$

Galois connection

$$T \subseteq \mathrm{I}(S) \quad \text{iff} \quad S \subseteq \mathrm{V}(T)\,.$$

- $\mathrm{V}(T)$ is always a closed cone of $\mathbb{R}^\kappa$.
- $\mathrm{I}(S)$ is always an ℓ-ideal of $\mathscr{F}_\kappa$ that is intersection of maximal ℓ-ideals.

Proposition

V *and* I *form a dual isomorphism between the poset of ℓ-ideals of $\mathscr{F}_\kappa$ that are intersections of maximal ℓ-ideals and the poset of closed cones in $\mathbb{R}^\kappa$.*

Theorem (Beynon 1974)

The category of semisimple Riesz spaces and Riesz space homomorphisms is dually equivalent to the category of closed cones in $\mathbb{R}^\kappa$ and piecewise linear maps between them.

On objects:

Let A be a semisimple Riesz space, then $A \cong \mathscr{F}_\kappa / J$, where J is an intersection of maximal ℓ-ideals of $\mathscr{F}_\kappa$. Then map

$$A \mapsto \mathsf{V}(J),$$

which is a closed cone in $\mathbb{R}^\kappa$.

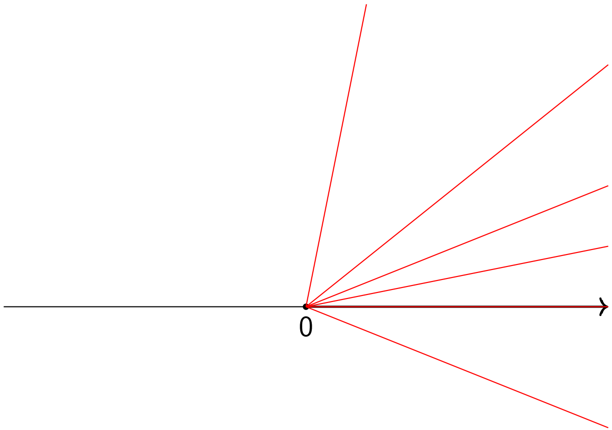
Let C be a closed cone in $\mathbb{R}^\kappa$. Then map

$$C \mapsto \mathsf{PWL}(C),$$

which is semisimple and isomorphic to $\mathscr{F}_\kappa / \mathsf{I}(C)$.

$\mathbb{R}$ (as a Riesz space) is dual to the semiline $\{x \in \mathbb{R} \mid x \geq 0\}$.

Indeed, $\mathbb{R} \cong \mathrm{PWL}(\{x \in \mathbb{R} \mid x \geq 0\})$. $\mathscr{F}_2 / \langle (x - y) \wedge y \wedge 0 \rangle$ is dual to $\{(x, y) \in \mathbb{R}^2 \mid 0 \leq y \leq x\}$.



Generalizing Baker-Beynon duality beyond semisimplicity

In the definition of the operators

$$\begin{aligned} \mathsf{V}(T) =& \{x \in \mathbb{R}^\kappa \mid t(x) = 0 \text{ for all } [t] \in T\} \text{ with } T \subseteq \mathscr{F}_\kappa \\ \mathsf{I}(S) =& \{[t] \in \mathscr{F}_\kappa \mid t(x) = 0 \text{ for all } x \in S\} \text{ with } S \subseteq \mathbb{R}^\kappa. \end{aligned}$$

we can replace $\mathbb{R}$ with any Riesz space A and still get a Galois connection.
In the definition of the operators

$$\begin{aligned} \mathsf{V}(T) =& \{x \in A^\kappa \mid t(x) = 0 \text{ for all } [t] \in T\} \text{ with } T \subseteq \mathscr{F}_\kappa \\ \mathsf{I}(S) =& \{[t] \in \mathscr{F}_\kappa \mid t(x) = 0 \text{ for all } x \in S\} \text{ with } S \subseteq A^\kappa. \end{aligned}$$

we can replace $\mathbb{R}$ with any Riesz space A and still get a Galois connection.

Caramello, Marra, and Spada (2021) observed that this can be done for any variety of algebras by replacing $\mathbb{R}$ with any algebra in that variety. They also show that this approach also works in a more categorical setting.

Our goal is to replace $\mathbb{R}$ with a Riesz space that guarantees more ℓ-ideals of $\mathscr{F}_\kappa$ to be fixpoints of IV.

It is not possible to obtain a Riesz space A such that for any κ the fixpoints of IV are all the ℓ-ideals of $\mathscr{F}_\kappa$.

Definition

An ℓ-ideal I is prime if $a \wedge b \in I$ implies $a \in I$ or $b \in I$.

Theorem

- *A/I is linearly ordered iff I is prime.*
- *Every ℓ-ideal is an intersection of prime ℓ-ideals.*
- *Every Riesz space is subdirect product of linearly ordered ones.*

Theorem (C., Lapenta, Spada)

Let α be a cardinal. There exists an ultrapower $\mathcal{U}$ of $\mathbb{R}$ in which all κ-generated (with $\kappa < \alpha$) linearly ordered Riesz spaces embed. In particular, when $\alpha = \omega$ we can take $\mathcal{U}$ to be any ultrapower of $\mathbb{R}$ over a nonprincipal ultrafilter of a countably infinite set.

Fix a cardinal α and an ultrapower $\mathcal{U}$ of $\mathbb{R}$ in which all κ-generated with $\kappa < \alpha$ linearly ordered Riesz spaces embed. κ will denote an arbitrary cardinal smaller than α.
We consider the operators:

$$\begin{aligned} \mathrm{V}(T) =& \{x \in \mathcal{U}^\kappa \mid t(x) = 0 \text{ for all } [t] \in T\} \text{ with } T \subseteq \mathscr{F}_\kappa \\ \mathrm{I}(S) =& \{[t] \in \mathscr{F}_\kappa \mid t(x) = 0 \text{ for all } x \in S\} \text{ with } S \subseteq \mathcal{U}^\kappa. \end{aligned}$$

Galois connection

$$T \subseteq \mathrm{I}(S) \quad \text{iff} \quad S \subseteq \mathrm{V}(T) \,.$$

- The fixpoints of $\mathrm{I}\,\mathrm{V}$ are exactly the ℓ-ideals of $\mathscr{F}_\kappa$.
- We call $S \subseteq \mathcal{U}^\kappa$ such that $S = \mathrm{V}\,\mathrm{I}(S)$ a generalized closed cone.

Proposition

V *and* I *establish a dual isomorphism between the poset of ℓ-ideals of $\mathscr{F}_\kappa$ and the poset of generalized closed cones in $\mathcal{U}^\kappa$.*

Definition

- We say that a map $\mathcal{U}^\kappa \to \mathcal{U}^\mu$ is definable if its components are defined by terms in the language of Riesz spaces.
- If $X \subseteq \mathcal{U}^\kappa$, we denote by $\mathrm{Def}(X)$ the set of definable maps $f : X \to \mathcal{U}$.

Theorem (C., Lapenta, Spada)

The category of κ-generated Riesz spaces (with $\kappa < \alpha$) and Riesz space homomorphisms is dually equivalent to the category of generalized closed cones in $\mathcal{U}^\kappa$ (with $\kappa < \alpha$) and definable maps.

On objects:

Let A be a κ-generated Riesz space, so $A \cong \mathscr{F}_\kappa / J$. Then map

$$A \mapsto \mathrm{V}(J),$$

which is a generalized closed cone in $\mathcal{U}^\kappa$.

Let C be a generalized closed cone in $\mathcal{U}^\kappa$. Then map

$$C \mapsto \mathrm{Def}(C),$$

which is isomorphic to $\mathscr{F}_\kappa / \mathrm{I}(C)$.

Consequences and applications

Proposition

- *The generalized closed cones in $\mathcal{U}^\kappa$ (together with $\varnothing$) form the closed of a topology on $\mathcal{U}^\kappa$. The closure of a nonempty $X \subseteq \mathcal{U}^\kappa$ is $\mathrm{V}\,\mathrm{I}(X)$.*
- *$\mathbb{R}^\kappa$ is a subset of $\mathcal{U}^\kappa$ and the closed subsets of $\mathbb{R}^\kappa$ with the subspace topology are exactly the closed cones (and $\varnothing$).*

$\mathscr{F}_\kappa$	$\mathbb{R}^\kappa$ (Baker-Beynon)	$\mathcal{U}^\kappa$ (gen. Baker-Beynon)
maximal ℓ-ideals	half-lines from the origin	closures of points of $\mathbb{R}^\kappa$ (except the origin)
intersections of maximal ℓ-ideals	closed cones	closures of nonempty subsets of $\mathbb{R}^\kappa$
prime ℓ-ideals		irreducible closed subsets $=$ closures of points of $\mathcal{U}^\kappa$ (except the origin)
ℓ-ideals		generalized closed cones

Definition

- Recall that an Riesz space is semisimple if the intersection of all its maximal ℓ-ideals is $\{0\}$.
- A Riesz space A is called Archimedean if for every $a, b \in A$, we have that $na \leq b$ for all $n \in \mathbb{N}$ implies $a \leq 0$.

- Semisemplicity always implies Archimedeanity.
- Archimedeanity implies semisimplicity in the presence of a strong order-unit (e.g., in the finitely generated setting).

Theorem

Let A be a Riesz space and $C \subseteq \mathcal{U}^\kappa$ its dual generalized closed cone. A is semisimple iff $C = \mathbb{V}\,\mathbb{I}(C \cap \mathbb{R}^\kappa)$, i.e. C is the closure of $C \cap \mathbb{R}^\kappa$ in $\mathcal{U}^\kappa$.

Note that $C \cap \mathbb{R}^\kappa$ is the closed cone in $\mathbb{R}^\kappa$ corresponding to A under Baker-Beynon duality.

For any natural number $n \geq 1$ let $\pi_n \colon \mathcal{U}^\omega \to \mathcal{U}^n$ be the map that sends $(x_i)_{i \in \omega}$ to $(x_1, \ldots, x_n)$.

Theorem

Let A be an ω-generated Riesz space and $C \subseteq \mathcal{U}^\omega$ its dual generalized closed cone.
Then A is archimedean iff

$$C = \bigcap_{n=1}^{\infty} \pi_n^{-1}[\mathrm{V}\,\mathrm{I}(\mathrm{V}\,\mathrm{I}(\pi_n[C]) \cap \mathbb{R}^n)],$$

where the subsets $\pi_n^{-1}[\mathrm{V}\,\mathrm{I}(\mathrm{V}\,\mathrm{I}(\pi_n[C]) \cap \mathbb{R}^n)]$ form a decreasing sequence of generalized closed cones in $\mathcal{U}^\omega$.

When $\kappa > \omega$, the decreasing sequence must be replaced by a downdirected family of generalized closed cones in $\mathcal{U}^\kappa$.

Embedding $\texttt{Spec}(\mathscr{F}_\kappa)$ into $\mathcal{U}^\kappa$

If A is a Riesz space, then $\mathtt{Spec}(A) = \{\text{prime } \ell\text{-ideals of } A\}$ is called the spectrum of A. It is naturally equipped with the Zariski topology generated by the closed subsets $\{P \in \mathtt{Spec}(A) \mid a \in P\}$, where a ranges in A.

If P is a prime ℓ-ideal of $\mathscr{F}_\kappa$, then $\mathsf{V}(P)$ is the closure of a point of $\mathcal{U}^\kappa$. For each prime ℓ-ideal P choose one such point and denote it by $\mathscr{E}(P)$.

Theorem

- $\mathscr{E} \colon \mathtt{Spec}(\mathscr{F}_\kappa) \to \mathcal{U}^\kappa$ *is a topological embedding.*
- *The posets of the open subsets of* $\mathtt{Spec}(\mathscr{F}_\kappa)$ *and of* $\mathcal{U}^\kappa \setminus \{O\}$ *are isomorphic.*

$\mathscr{E}\colon \mathtt{Spec}(\mathscr{F}_\kappa) \to \mathcal{U}^\kappa$ can be thought of as a coordinatization of $\mathtt{Spec}(\mathscr{F}_\kappa)$ with coordinates in $\mathcal{U}$.

- By the correspondence theorem, if $A \cong \mathscr{F}_\kappa / J$, then we can think of $\mathtt{Spec}(A)$ as a subspace of $\mathtt{Spec}(\mathscr{F}_\kappa)$.
- $\mathscr{E}$ restricts to an embedding of $\mathtt{Spec}(A)$ into $\mathcal{U}^\kappa$ whose image is $\mathscr{E}[\mathtt{Spec}(\mathscr{F}_\kappa)] \cap \mathsf{V}(J)$.

While the spectrum as a topological space is not sufficient to recover the original Riesz space, the coordinatization is enough:

Theorem

$A \cong \mathrm{Def}(\mathscr{E}[\mathtt{Spec}(A)])$ *for any Riesz space A.*

$\mathcal{U}$ as the non-standard line

Let $\alpha = \omega$ and assume that $\mathcal{U}$ is an ultrapower of $\mathbb{R}$ defined as $\mathcal{U} = \mathbb{R}^{\mathbb{N}}/\mathcal{F}$ with $\mathcal{F}$ a nonprincipal ultrafilter of $\mathcal{P}(\mathbb{N})$.

We can think of $\mathcal{U}$ as a non-standard line. $\mathcal{U}$ is a linearly ordered field containing (a copy of) $\mathbb{R}$. The elements of $\mathcal{U}$ are called hyperreal numbers.

- $x \in \mathcal{U}$ is infinitesimal if $|x| \leq r$ for every $0 < r \in \mathbb{R}$.
- $x \in \mathcal{U}$ is unlimited if $|x| \geq r$ for every $0 < r \in \mathbb{R}$.

Working with an ultrapower allows us to define the enlargement ${}^*A \subseteq \mathcal{U}^n$ of a subset $A \subseteq \mathbb{R}^n$, as well as the enlargement ${}^*f : \mathcal{U}^n \to \mathcal{U}$ of a function $f : \mathbb{R}^n \to \mathbb{R}$.

Theorem (Transfer principle)

Let φ be a first-order sentence. Then φ is true in $\mathbb{R}$ iff ${}^*\varphi$ is true in $\mathcal{U}$.

- $A \subseteq {}^*A$.
- If A is finite, then $A = {}^*A$.
- ${}^*\mathbb{Q}$ contains both nonzero infinitesimal and unlimited numbers.

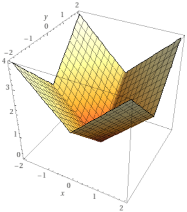
Let $g\colon \mathcal{U}^n \to \mathcal{U}$ be definable, i.e. there is a term t such that $g(x) = t(x)$ for all $x \in \mathcal{U}^n$. If $f\colon \mathbb{R}^n \to \mathbb{R}$ is the piecewise linear function defined by the same term, then $g = {}^*f$.

Proposition

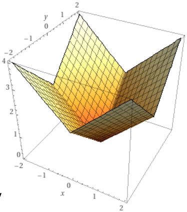
Let $C \subseteq \mathcal{U}^n$ be a generalized closed cone. Then
$\mathrm{Def}(C) = \{({}^*f)_{|C} \mid f\colon \mathbb{R}^n \to \mathbb{R}$ *piecewise linear*$\}$.

The graph of ${}^*f\colon \mathcal{U}^n \to \mathcal{U}$ is just the enlargement of the graph of f.

$f\colon \mathbb{R}^2 \to \mathbb{R}$



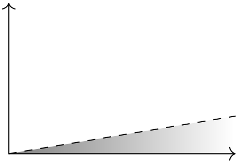
${}^*f\colon \mathcal{U}^2 \to \mathcal{U}$



Definable functions naturally generalize piecewise linear functions.

Let $\mathbb{R} \overrightarrow{\times} \mathbb{R}$. Then its dual generalized closed cone is

$$C = \{(x, y) \in \mathcal{U}^2 \mid x > 0,\ y \geq 0,\ \text{and } y/x \text{ is infinitesimal}\} \cup \{(0, 0)\}.$$



So,

$$\begin{aligned}\mathbb{R} \overrightarrow{\times} \mathbb{R} \cong \mathrm{Def}(C) &= \{({}^*f)_{|C} \mid f \colon \mathbb{R}^2 \to \mathbb{R} \text{ piecewise linear}\} \\ &= \{({}^*f)_{|C} \mid f \colon \mathbb{R}^2 \to \mathbb{R} \text{ linear}\}.\end{aligned}$$

Indexes and prime ideals

We have seen that prime ℓ-ideals of $\mathscr{F}_n$ (and hence n-generated linearly ordered Riesz spaces) correspond to the closures of the points of $\mathcal{U}^n$.

We want to understand how these subsets of $\mathcal{U}^n$ look like.

Theorem (Orthogonal decomposition)

If $x \in \mathcal{U}^n$, then $x = \alpha_1 v_1 + \cdots + \alpha_k v_k$ where $\alpha_1, \ldots, \alpha_k \in \mathcal{U}$ are positive, α_{i+1}/α_i is infinitesimal for each $i < k$, and $v_1, \ldots, v_k \in \mathbb{R}^n$ are orthonormal vectors. Furthermore, this decomposition is unique.

Definition

- We call a finite sequence $(v_1, \ldots, v_k)$ of orthonormal vectors in $\mathbb{R}^n$ an index.
- We denote by $\iota(x)$ the index $(v_1, \ldots, v_k)$ made of the vectors appearing in the orthogonal decomposition of $x \in \mathcal{U}^n$.
- Let $\mathbf{v}, \mathbf{w}$ be two indexes. We write $\mathbf{v} \leq \mathbf{w}$ when $\mathbf{v}$ is a truncation of $\mathbf{w}$, i.e. $\mathbf{v} = (v_1, \ldots, v_h)$ and $\mathbf{w} = (v_1, \ldots, v_k)$ for $h \leq k$.

Definition

If $\mathbf{v}$ is an index, let $\mathrm{Cone}(\mathbf{v}) := \{y \in \mathcal{U}^n \mid \iota(y) \leq \mathbf{v}\}$.

Theorem

The closure of x in $\mathcal{U}^n$ is $\mathrm{Cone}(\iota(x))$.

Proposition

If $x \in \mathcal{U}^n$, then

$$\begin{aligned}\mathrm{Def}(\mathrm{Cone}(\iota(x))) &\cong \{{}^*f(x) \in \mathcal{U} \mid f : \mathbb{R}^n \to \mathbb{R} \textit{ piecewise linear}\} \\ &= \{{}^*f(x) \in \mathcal{U} \mid f : \mathbb{R}^n \to \mathbb{R} \textit{ linear}\}.\end{aligned}$$

So, each *n*-generated linearly ordered Riesz spaces is of the form $\{{}^*f(x) \mid f : \mathbb{R}^n \to \mathbb{R} \text{ linear}\}$ for some $x \in \mathcal{U}^n$.

Let $\varepsilon \in \mathcal{U}$ be a positive infinitesimal and $x = (1, \varepsilon)$. Then

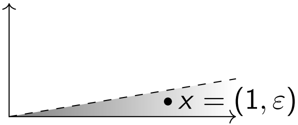
$$x = 1(1,0) + \varepsilon(0,1)$$

is the orthogonal decomposition of x. Thus, $\iota(x) = (v_1, v_2)$ with $v_1 = (1,0)$ and $v_2 = (0,1)$. We have

$$y \in \mathrm{Cone}(\iota(x)) \quad \text{iff} \quad \begin{aligned} & y = O, \text{ or} \\ & y = \alpha_1(1,0) \text{ (orthogonal decomposition), or} \\ & y = \alpha_1(1,0) + \alpha_2(0,1) \text{ (orthogonal decomposition)} \end{aligned}$$

Then $\mathrm{Cone}(\iota(x))$, i.e. the closure of x in $\mathcal{U}^2$ is

$$\{(\alpha_1, \alpha_2) \in \mathcal{U}^2 \mid \alpha_1 > 0,\ \alpha_2 \geq 0 \text{ and } \alpha_2/\alpha_1 \text{ is infinitesimal}\} \cup \{O\}.$$



The dual Riesz space is $\mathbb{R} \overrightarrow{\times} \mathbb{R}$. Indeed,

$$\begin{aligned} \mathrm{Def}(\mathrm{Cone}(\iota(x))) &\cong \{{}^*f(1,\varepsilon) \mid f \colon \mathbb{R}^n \to \mathbb{R} \text{ linear}\} \\ &= \{a + b\varepsilon \in \mathcal{U} \mid a, b \in \mathbb{R}\} \cong \mathbb{R} \overrightarrow{\times} \mathbb{R}. \end{aligned}$$

Theorem

$\mathrm{I} \circ \mathrm{Cone} \colon \mathbf{v} \mapsto \mathrm{I}(\mathrm{Cone}(\mathbf{v}))$ *induces an order-isomorphism between the set of indexes ordered by truncation and* $\mathtt{Spec}(\mathscr{F}_n)$ *ordered by reverse inclusion.*

If $\mathbf{v} = (v_1, \ldots, v_k)$ is an index, then we call a subset of $\mathbb{R}^n$ a **v**-cone if it is the positive span of $\{\sum_{i=1}^{h} r_i v_i \mid h \leq k\}$ for some $0 < r_1, \ldots, r_k \in \mathbb{R}$.

Theorem

$\mathrm{Cone}(\mathbf{v})$ *is the intersection of the enlargements of all the* $\mathbf{v}$*-cones.*

Corollary (Panti 1999)

Every prime ℓ*-ideal of* $\mathrm{PWL}(\mathbb{R}^n)$ *is of the form*

$$\{f \in \mathrm{PWL}(\mathbb{R}^n) \mid f \text{ vanishes on some } \mathbf{v}\text{-cone}\}$$

for some index $\mathbf{v}$*.*

Main idea of the proof: Let $f \in \mathrm{PWL}(\mathbb{R}^n)$. By the transfer principle, ${}^*f$ vanishes on $\mathrm{Cone}(\mathbf{v})$ iff f vanishes on some $\mathbf{v}$-cone.

Recall: if we map $P \in \mathtt{Spec}(\mathscr{F}_n)$ to a point $x \in \mathcal{U}^n$ such that $\mathrm{V}(P)$ is the closure of x, then we get an embedding $\mathscr{E} \colon \mathtt{Spec}(\mathscr{F}_n) \to \mathcal{U}^n$.

Indexes allow us to choose x for every P in a canonical way (modulo fixing a positive infinitesimal $\varepsilon \in \mathcal{U}$).

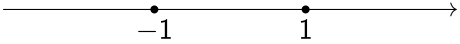
If $P \in \mathtt{Spec}(\mathscr{F}_n)$, then there is a unique index $\mathbf{v} = (v_1, \dots, v_k)$ such that $\mathrm{V}(P) = \mathrm{Cone}(\mathbf{v})$. Define

$$\mathscr{E}(P) := v_1 + \varepsilon v_2 + \cdots + \varepsilon^{k-1} v_k.$$

If P is a maximal ℓ-ideal, then the corresponding index consists of a single vector $\mathbf{v} = (v_1)$. Therefore, $\mathscr{E}(P) \in \mathbb{R}^n$.

$\mathtt{Spec}(\mathscr{F}_1)$

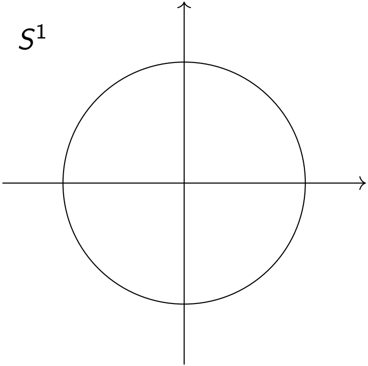
We have $\mathscr{E}[\mathtt{Spec}(\mathscr{F}_1)] = \{-1, 1\} \subseteq \mathcal{U}$.



Note that $\mathtt{Spec}(\mathscr{F}_1) = \mathtt{MaxSpec}(\mathscr{F}_1)$.

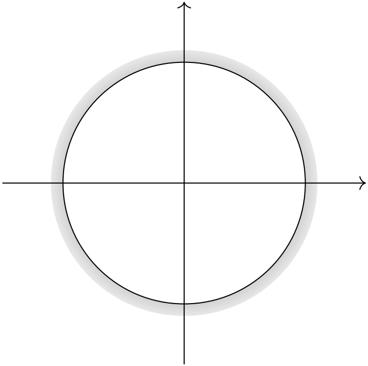
$\texttt{Spec}(\mathscr{F}_2)$

We have $\mathscr{E}[\texttt{MaxSpec}(\mathscr{F}_2)] = S^1 \subseteq \mathbb{R}^2 \subseteq \mathcal{U}^2$.



$\mathtt{Spec}(\mathscr{F}_2)$

We have $\mathscr{E}[\mathtt{Spec}(\mathscr{F}_2)] \subseteq \mathcal{U}^2$ consists of points infinitesimally close to S^1.



$\mathtt{Spec}(\mathscr{F}_2)$

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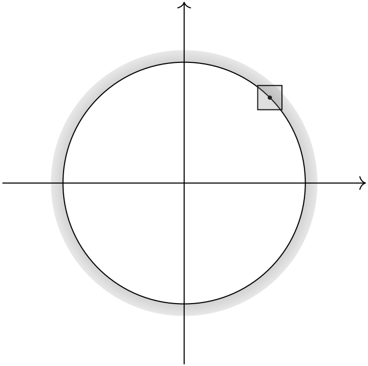

Spec($\mathscr{F}_2$)

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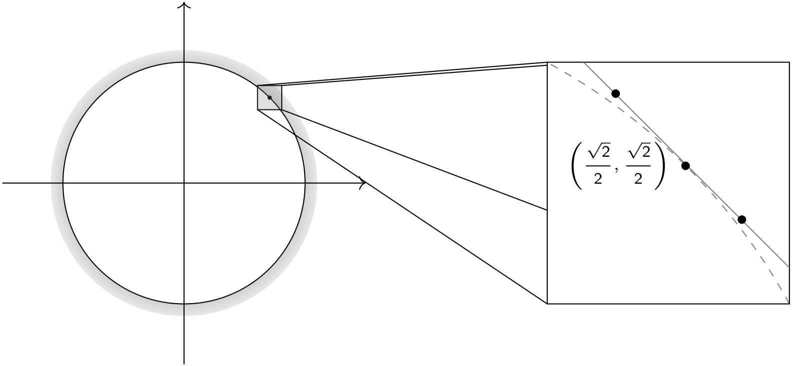

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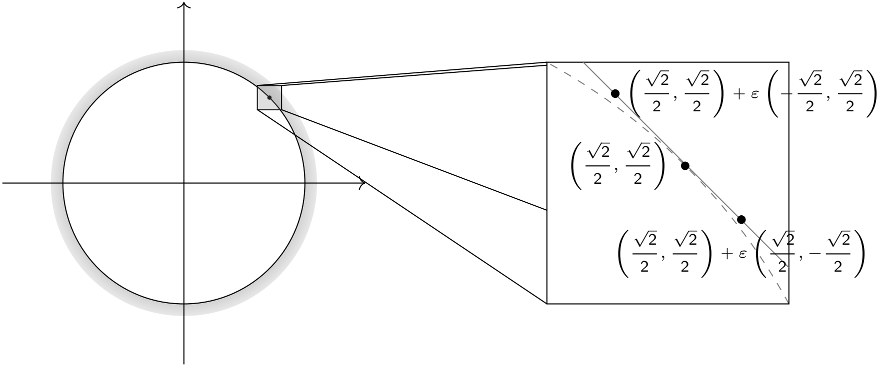
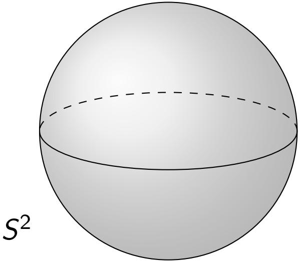


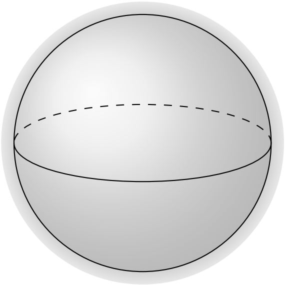
$\mathtt{Spec}(\mathscr{F}_3)$

We have $\mathscr{E}[\mathtt{MaxSpec}(\mathscr{F}_3)] = S^2 \subseteq \mathbb{R}^3 \subseteq \mathcal{U}^3$.



$\mathtt{Spec}(\mathscr{F}_3)$

We have $\mathscr{E}[\mathtt{Spec}(\mathscr{F}_3)] \subseteq \mathcal{U}^3$ consists of points infinitesimally close to S^2.



$\mathtt{Spec}(\mathscr{F}_3)$

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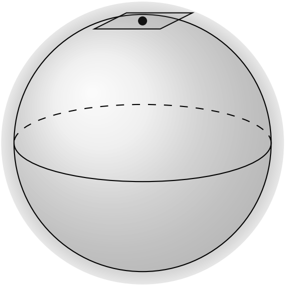

$\mathrm{Spec}(\mathscr{F}_3)$

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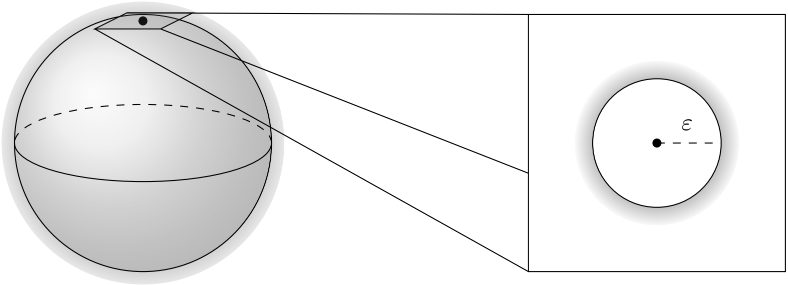

THANK YOU!