Extending the Blok-Esakia Theorem to the monadic setting

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The Blok-Esakia Theorem and its algebraic proof

Intuitionistic logic

- logic of constructive mathematics
- rejects the law of excluded middle $p \vee \neg p$
- IPC denotes the intuitionistic propositional calculus

Modal logic

- enriches classical logic with modalities
- S4 is a propositional modal logic obtained by enriching propositional classical logic with a modality $\Box$ subject to some axioms. It is the modal logic of quasi-ordered Kripke frames.

The Gödel (or Gödel-McKinsey-Tarski) translation allows us to think of IPC as a fragment of S4.

The Gödel translation (1933)

$$\begin{aligned}
\mathsf{T}(\bot) &= \bot \\
\mathsf{T}(p) &= \Box p \\
\mathsf{T}(\varphi \wedge \psi) &= \mathsf{T}(\varphi) \wedge \mathsf{T}(\psi) \\
\mathsf{T}(\varphi \vee \psi) &= \mathsf{T}(\varphi) \vee \mathsf{T}(\psi) \\
\mathsf{T}(\varphi \to \psi) &= \Box(\neg \mathsf{T}(\varphi) \vee \mathsf{T}(\psi))
\end{aligned}$$

Gödel observed that if $\mathsf{IPC} \vdash \varphi$, then $\mathsf{S4} \vdash T(\varphi)$, and conjectured that also the converse holds.

Theorem (McKinsey-Tarski 1948)

T *embeds* IPC *faithfully into* S4, *i.e.*

$$\mathsf{IPC} \vdash \varphi \quad \textit{iff} \quad \mathsf{S4} \vdash T(\varphi)$$

for any formula φ.

Dummett and Lemmon in the 1950s started studying the Gödel translation between superintuitionistic logics, i.e. extensions of IPC, and (normal) extensions of S4.

Definition

Let L be a superintuitionistic logic and M an extension of S4. We call L the intuitionistic fragment of M and M a modal companion of L if

$$\mathsf{L} \vdash \varphi \qquad \text{iff} \qquad \mathsf{M} \vdash \mathsf{T}(\varphi)$$

for any intuitionistic formula φ.

They showed that each superintuitionistic logic has at least one modal companion.

The least modal companion of IPC is S4.

Definition

Let $\mathsf{Grz} := \mathsf{S4} + \Box(\Box(p \to \Box p) \to p) \to p$

Grzegorczyk showed that IPC faithfully embeds into Grz.

Theorem (Grzegorczyk 1967)

Grz *is a modal companion of* IPC*.*

Esakia showed that Grz is the largest extension of S4 with this property.

Theorem (Esakia's Theorem 1976)

Grz *is the greatest modal companion of* IPC*.*

Maksimova and Rybakov introduced the mappings ρ, τ, and σ.

Definition

Let M be an extension of S4 and L a superintionistic logic.

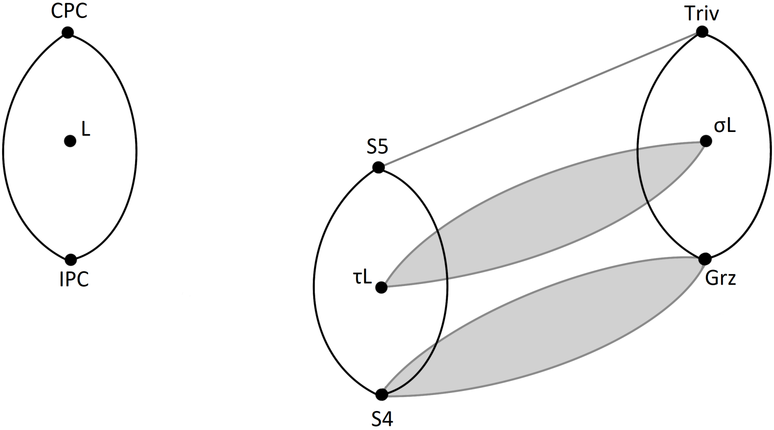
- $\rho\mathsf{M} := \{\varphi \mid \mathsf{M} \vdash \mathsf{T}(\varphi)\}$, the intuitionistic fragment of M.
- $\tau\mathsf{L} := \mathsf{S4} + \{\mathsf{T}(\varphi) \mid \mathsf{L} \vdash \varphi\}$, the least modal companion of L.

Theorem (Maksimova and Rybakov 1974)

Let L *be a superintuitionistic logic. Then* L *has a greatest modal companion* $\sigma\mathsf{L}$.

Theorem (Blok-Esakia 1976)

- $\sigma\mathsf{L} = \mathsf{Grz} + \{\mathsf{T}(\varphi) \mid \mathsf{L} \vdash \varphi\}$.
- σ *is an isomorphism between the lattice of superintuitionistic logics and the lattice of extensions of* Grz*, whose inverse is* ρ.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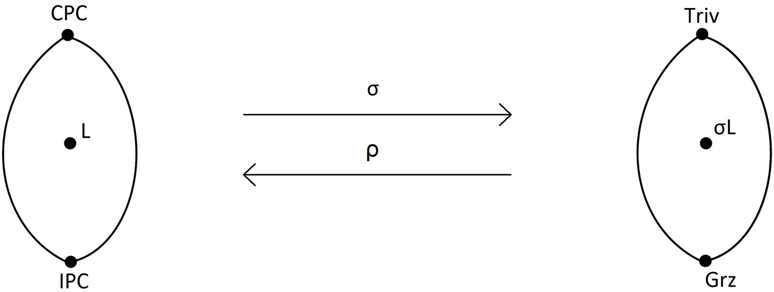
CPC
L
IPC
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Grz

Definition

A Heyting algebra H is a distributive lattice equipped with a binary operation $\rightarrow$ such that for every $a, b, c \in H$:

$$a \wedge b \leq c \iff a \leq b \rightarrow c.$$

Theorem (algebraic semantics for IPC)

$\mathsf{IPC} \vdash \varphi$ *iff* $H \vDash \varphi$ *for every Heyting algebra* H.

Definition

An S4-algebra B is a boolean algebra equipped with a unary operator $\Box$ such that such that for every $a, b \in B$:

$$\Box 1 = 1, \qquad \Box(a \wedge b) = \Box a \wedge \Box b, \qquad \Box a \leq a, \qquad \Box a \leq \Box\Box a.$$

Theorem (algebraic semantics for S4)

$\mathsf{S4} \vdash \varphi$ *iff* $B \vDash \varphi$ *for every* S4*-algebra* B.

Definition

- If B is an S4-algebra, then $\mathcal{O}(B) := \{b \in B \mid \Box b = b\}$ is a Heyting algebra with $a \to b := \Box(\neg a \vee b)$.
- If H is a Heyting algebra, then the free boolean extension $\mathcal{B}(H)$ of H with the operator

$$\Box\left(\bigwedge_{1}^{n}(\neg a_i \vee b_i)\right) := \bigwedge_{1}^{n}(a_i \to b_i)$$

  is an S4-algebra. In fact, a Grz-algebra.

Theorem

- *$\mathcal{O}(B) \vDash \varphi$ iff $B \vDash T(\varphi)$.*
- *If H is an Heyting algebra, then $\mathcal{O}\mathcal{B}(H) \cong H$.*
- *If B is an S4-algebra, then $\mathcal{B}\mathcal{O}(B)$ embeds into B.*

- The category of Heyting algebras is dually equivalent to the category of Esakia spaces.
- The category of S4-algebras is dually equivalent to the category of S4-spaces.

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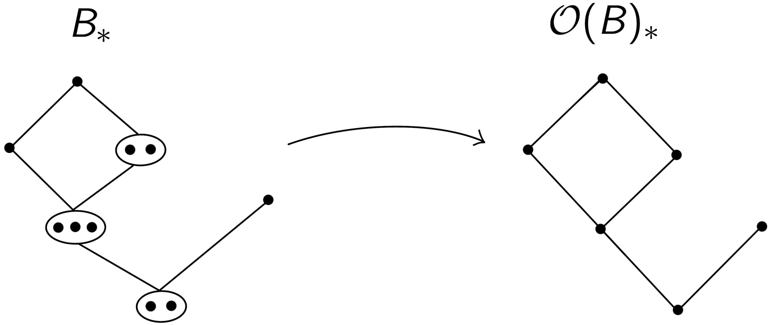
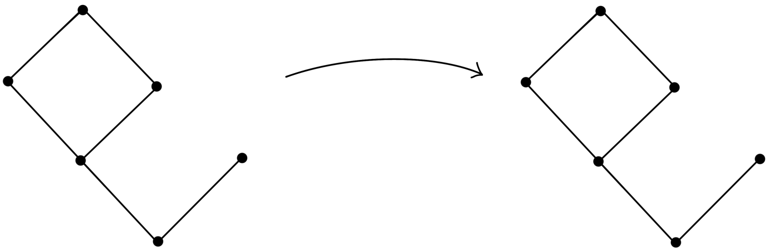
$\mathcal{O}$ corresponds to taking the skeleton of an S4-space.



- The category of Heyting algebras is dually equivalent to the category of Esakia spaces.
- The category of S4-algebras is dually equivalent to the category of S4-spaces.

$\mathcal{B}$ corresponds to thinking of an Esakia space as an S4-space.

H_* $\mathcal{B}(H)_*$



Varieties of Heyting algebras and S4-algebras correspond to superintuitionistic logics and extensions of S4, respectively.

If $\mathbb{K}$ is a class of S4-algebras, let $\mathcal{O}(\mathbb{K}) := \{\mathcal{O}(B) \mid B \in \mathbb{K}\}$

Theorem

- *$\mathcal{O}$ commutes with* H, S, *and* P*.*
- *If $\mathbb{V}$ is a variety of* S4*-algebras corresponding to an extension* M *of* S4*, then $\mathcal{O}(\mathbb{V})$ is a variety of Heyting algebras and corresponds to ρ*M*.*

If $\mathbb{K}$ is a class of Heyting algebras, let $\mathcal{B}(\mathbb{K}) := \{\mathcal{B}(H) \mid H \in \mathbb{K}\}$.

Proposition

$\mathcal{B}$ commutes with H and S, but it does not commute with infinite products.

Define $\mathcal{B}^*(\mathbb{K}) := \mathsf{HSP}(\mathcal{B}(\mathbb{K}))$.

Theorem (Maksimova and Rybakov 1974)

Let $\mathbb{V}$ be a variety of Heyting algebras.

- $\mathcal{B}^*(\mathbb{V})$ *is a variety of* Grz-*algebras.*
- $\mathcal{O}\mathcal{B}^*(\mathbb{V}) = \mathbb{V}$. *Consequence: ρ is onto.*
- $\mathcal{B}^*(\mathbb{V})$ *is the smallest variety $\mathbb{W}$ of* S4-*algebras such that $\mathcal{O}(\mathbb{W}) = \mathbb{V}$. Consequence: if $\mathbb{V}$ corresponds to* L, *then $\mathcal{B}^*(\mathbb{V})$ corresponds to the largest modal companion σ*L *of* L.

The following is the key to the proof of the Blok-Esakia Theorem.

Theorem (Blok's Lemma 1976)

- *If B is a* Grz-*algebra, then B and $\mathcal{B}\mathcal{O}(B)$ generate the same variety.*
- *If $\mathbb{W}$ is a variety of* Grz-*algebras, then $\mathcal{B}^*\mathcal{O}(\mathbb{W}) = \mathbb{W}$.*

To the monadic setting

Rasiowa and Sikorski extended the Gödel translation to the predicate setting as follows:

$$\begin{aligned} \mathsf{T}(\forall x\varphi) &= \Box\forall x\mathsf{T}(\varphi) \\ \mathsf{T}(\exists x\varphi) &= \exists x\mathsf{T}(\varphi) \end{aligned}$$

Theorem (Rasiowa-Sikorski 1953)

T *faithfully embeds the intuitionistic predicate calculus* IQC *into the predicate* S4 *logic* QS4, *i.e.*

$$\mathsf{IQC} \vdash \varphi \qquad \textit{iff} \qquad \mathsf{QS4} \vdash T(\varphi)$$

for any formula φ.

Definition

The monadic fragment of a predicate logic L is the set of theorems of L in one fixed variable containing only unary predicate symbols.

Example

$$\forall x(P(x) \to \exists x Q(x))$$

$$\forall(p \to \exists q)$$

Therefore, monadic fragments can be treated like propositional modal logics with modalities $\forall$, $\exists$.

Definition

- MIPC is the monadic fragment of the intuitionistic predicate calculus IQC.
- MS4 is the monadic fragment of the predicate S4 logic QS4.

The predicate Gödel translation restricts to a full and faithful translation of MIPC into MS4.

$$\begin{aligned} \mathsf{T}(\forall\varphi) &= \Box\forall\mathsf{T}(\varphi) \\ \mathsf{T}(\exists\varphi) &= \exists\mathsf{T}(\varphi) \end{aligned}$$

Let M be an extension of MS4 and L an extension of MIPC. The intuitionistic fragment of M and modal companions of L are defined similarly to the propositional case. We also define:

- $\rho\mathsf{M} := \{\varphi \mid \mathsf{M} \vdash \mathsf{T}(\varphi)\}$, the intuitionistic fragment of M.
- $\tau\mathsf{L} := \mathsf{MS4} + \{\mathsf{T}(\varphi) \mid \mathsf{L} \vdash \varphi\}$.
- $\sigma\mathsf{L} := \mathsf{MGrz} + \{\mathsf{T}(\varphi) \mid \mathsf{L} \vdash \varphi\}$.

where MGrz is $\mathsf{MS4} + \mathsf{grz}$.

Definition

A monadic Heyting algebra H is a Heyting algebra equipped with two unary operators $\forall, \exists$ satisfying for every $a, b \in H$:

$$\forall(a \wedge b) = \forall a \wedge \forall b \qquad \exists(a \vee b) = \exists a \vee \exists b$$

$$\forall 1 = 1 \qquad \exists 0 = 0$$

$$\forall a \leq a \qquad a \leq \exists a$$

$$\forall \exists a = \exists a \qquad \exists \forall a = \forall a$$

$$\exists(\exists a \wedge b) = \exists a \wedge \exists b$$

A monadic S4-algebra is an S4-algebra equipped with a unary operator $\forall$ satisfying for any $a, b \in B$:

$$\forall(a \wedge b) = \forall a \wedge \forall b \qquad \forall 1 = 1$$

$$\forall a \leq a \qquad a \leq \forall \neg \forall \neg a$$

$$\Box \forall a \leq \forall \Box a$$

Theorem (Algebraic semantics)

- MIPC $\vdash \varphi$ *iff* $H \vDash \varphi$ *for every monadic Heyting algebra* H.
- MS4 $\vdash \varphi$ *iff* $B \vDash \varphi$ *for every* MS4-*algebra* B.
- *Varieties of monadic Heyting algebras correspond to extensions of* MIPC.
- *Varieties of* MS4-*algebras correspond to extensions of* MS4.

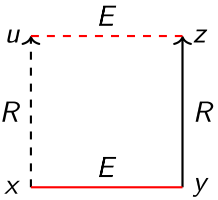
Definition

- If B is an MS4-algebra, then $H = \mathcal{O}(B)$ is a monadic Heyting algebra with $\forall_H a = \Box \forall_B a$ and $\exists_H a = \exists_B a$.
- (Fischer Servi 1977) If H is a **finite** monadic Heyting algebra, then the free boolean extension $\mathcal{B}(H)$ can be equipped with a structure of MS4-algebra. It is always a MGrz-algebra.

- The category of monadic Heyting algebras is dually equivalent to the category of descriptive MIPC-frames.
- The category of MS4-algebras is dually equivalent to the category of descriptive MS4-frames.

A descriptive MIPC-frame (MS4-frame) is an Esakia space (S4-space) (X, R) equipped with an additional equivalence relation E such that:

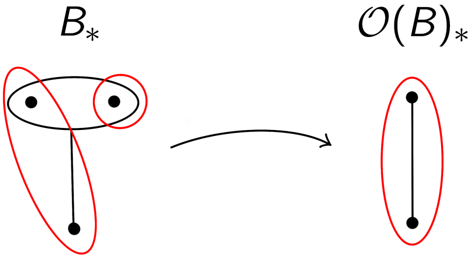
xEy and yRz imply there is $u \in X$ s.t. xRu and uEz.



and E satisfies some topological conditions.

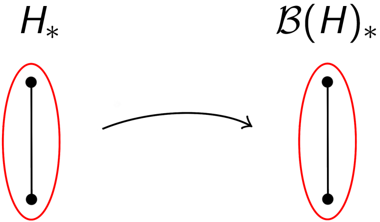
- The category of monadic Heyting algebras is dually equivalent to the category of descriptive MIPC-frames.
- The category of MS4-algebras is dually equivalent to the category of descriptive MS4-frames.

$\mathcal{O}$ corresponds to taking the skeleton of a descriptive MS4-frame.



- The category of monadic Heyting algebras is dually equivalent to the category of descriptive MIPC-frames.
- The category of MS4-algebras is dually equivalent to the category of descriptive MS4-frames.

$\mathcal{B}$ corresponds to thinking of a **finite** descriptive MIPC-frame as a descriptive MS4-frame.



- The category of monadic Heyting algebras is dually equivalent to the category of descriptive MIPC-frames.
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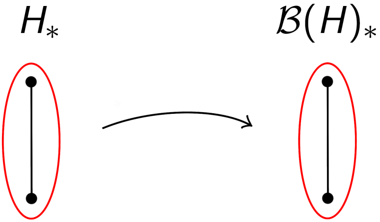
$\mathcal{B}$ corresponds to thinking of a **finite** descriptive MIPC-frame as a descriptive MS4-frame.



Problem

An infinite descriptive MIPC-frame is not always a descriptive MS4-frame.

If $\mathbb{K}$ is a class of MS4-algebras, let $\mathcal{O}(\mathbb{K}) := \{\mathcal{O}(B) \mid B \in \mathbb{K}\}$.

Theorem

- *$\mathcal{O}$ commutes with* H *and* P.
- $\mathcal{O}\mathrm{S}(\mathbb{K}) \subseteq \mathrm{S}\mathcal{O}(\mathbb{K})$, *the other inclusion is not true in general.*
- *If $\mathbb{V}$ is a variety of* MS4*-algebras, then* $\mathrm{S}\mathcal{O}(\mathbb{V})$ *is the variety generated by $\mathcal{O}(\mathbb{V})$.*

Problem

If $\mathbb{V}$ is a variety of MS4 algebras, then $\mathcal{O}(\mathbb{V})$ is not necessarily a variety.

Theorem

Let $\mathbb{V}$ be a variety of MS4*-algebras corresponding to an extension* M *of* MS4*. Then* $\mathrm{S}\mathcal{O}(\mathbb{V})$ *is the variety of monadic Heyting algebras corresponding to* ρM.

Failure of the Blok-Esakia and Esakia's Theorems in the monadic setting

Theorem

- $\mathsf{S}\mathcal{O}$ *preserves joins of varieties.*
- $\mathsf{S}\mathcal{O}$ *does not preserve binary intersections of varieties.*
- $\mathsf{S}\mathcal{O}$ *is not one-to-one on varieties of* MGrz*-algebras.*

Corollary

In the monadic setting:

- ρ *preserves arbitrary intersections, but not binary joins.*
- τ *and* σ *preserve binary intersections and arbitrary joins.*
- ρ *is not 1-1 on extensions of* MGrz*.*

In the propositional setting:

- All ρ, τ, and σ preserve arbitrary intersections and joins.
- ρ is 1-1 on extensions of Grz.

Theorem (Failure of the monadic Blok-Esakia Theorem)

σ is not onto, and hence is not an isomorphism.

Sketch of the proof:

Let ρ' be the restriction of ρ to the lattice of extensions of MGrz.

σ is left adjoint to ρ'. Since ρ' is not 1-1, σ cannot be onto.

Three equivalent open problems

- Does every extension of MIPC have a modal companion?
- Is ρ onto?
- Is τ one-to-one?

The finite model property of MIPC (Bull 1965, Ono 1977, Fischer Servi 1978) yields:

Theorem

MGrz *is a modal companion of* MIPC.

Does Esakia's Theorem generalize to MIPC?

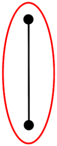
- Is MGrz the largest modal companion of MIPC?
- If not, is there a largest modal companion of MIPC?

Let Kur $:= \mathsf{MIPC} + \forall\neg\neg p \to \neg\neg\forall p$ be the monadic Kuroda logic.

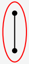
Theorem (Esakia)

Let H be a monadic Heyting algebra.
$H \vDash \mathsf{Kur}$ *iff* $E[\max H_*] \subseteq \max H_*$.

The dual of the following frame does not validate Kur.



Theorem (Esakia-Bezhanishvili 1998)

Kur *is the splitting logic axiomatized by the Jankov formula* $\mathcal{J}($  $)$.

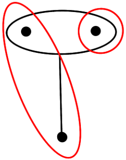
Let LKur be the extension of MS4 axiomatized by the following formula

$$\Box\forall(\Diamond p \wedge \exists p \wedge \exists\neg p \wedge (p \to \Box p)) \to p.$$

Let qmax X the set of elements x such that xRy implies yRx.

Theorem

Let B be an MS4*-algebra. Then $B \vDash$* LKur *iff for every $x \in \operatorname{qmax} B_*$, there is y such that xRy and $E[y] \subseteq \operatorname{qmax} B_*$.*



Theorem

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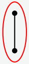
$\mathcal{J}($  $)$.

Theorem

- LKur *is a modal companion of* MIPC.
- Kur $\subseteq \rho(\mathsf{MGrz} \vee \mathsf{LKur})$. *So,* MGrz $\vee$ LKur *is not a modal companion of* MIPC.

Recall that MGrz is a modal companion of MIPC.

Theorem (Failure of Esakia's Theorem for MIPC)

There is no greatest modal companion of MIPC.

Open problems

By Zorn's Lemma there are maximal modal companions of MIPC. How many? Is MGrz maximal?

THANK YOU!