Extending the Blok-Esakia Theorem to the monadic setting

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The Blok-Esakia Theorem

Intuitionistic logic

- Logic of constructive mathematics.
- Does not assume the law of excluded middle $p \vee \neg p$.
- IPC denotes the intuitionistic propositional calculus.

Modal logic

- Enriches classical logic with modalities.
- The propositional modal logic S4 is obtained by adding to the classical propositional calculus a unary modality $\Box$ subject to certain axioms and inference rules.
- S4 is the modal logic of quasi-ordered Kripke frames.

The Gödel (or Gödel-McKinsey-Tarski) translation allows us to think of IPC as a fragment of S4.

The Gödel translation (1933)

$$\begin{array}{rcl} \mathrm{T}(\bot) & = & \bot \\ \mathrm{T}(p) & = & \Box p \\ \mathrm{T}(\varphi \wedge \psi) & = & \mathrm{T}(\varphi) \wedge \mathrm{T}(\psi) \\ \mathrm{T}(\varphi \vee \psi) & = & \mathrm{T}(\varphi) \vee \mathrm{T}(\psi) \\ \mathrm{T}(\varphi \to \psi) & = & \Box(\neg \mathrm{T}(\varphi) \vee \mathrm{T}(\psi)) \end{array}$$

Gödel observed that if $\mathrm{IPC} \vdash \varphi$, then $\mathrm{S4} \vdash T(\varphi)$, and conjectured that also the converse holds.

Theorem (McKinsey-Tarski 1948)

T *embeds* IPC *faithfully into* S4, *i.e.*

$$\mathrm{IPC} \vdash \varphi \qquad \textit{iff} \qquad \mathrm{S4} \vdash T(\varphi)$$

for any formula φ.

Dummett and Lemmon in the 1950s started studying the Gödel translation between superintuitionistic logics (i.e., extensions of IPC) and (normal) extensions of S4.

Definition

Let L be a superintuitionistic logic and M an extension of S4. We call L the intuitionistic fragment of M and M a modal companion of L if

$$\mathsf{L} \vdash \varphi \qquad \text{iff} \qquad \mathsf{M} \vdash \mathsf{T}(\varphi)$$

for any intuitionistic formula φ.

Theorem (Dummett and Lemmon 1959)

Each superintuitionistic logic L *has a least modal companion given by* $\mathsf{S4} + \{\mathsf{T}(\varphi) \mid \mathsf{L} \vdash \varphi\}$.

The least modal companion of IPC is S4.

Definition

Let $\mathsf{Grz} := \mathsf{S4} + \Box(\Box(p \to \Box p) \to p) \to p$

Grzegorczyk showed that IPC faithfully embeds into Grz.

Theorem (Grzegorczyk 1967)

Grz *is a modal companion of* IPC.

Esakia showed that Grz is the largest extension of S4 with this property.

Theorem (Esakia's Theorem 1976)

Grz *is the greatest modal companion of* IPC.

Maksimova and Rybakov introduced the mappings ρ, τ, and σ.

Definition

Let M be an extension of S4 and L a superintuitionistic logic.

- $\rho\mathsf{M} := \{\varphi \mid \mathsf{M} \vdash \mathsf{T}(\varphi)\}$, the intuitionistic fragment of M.
- $\tau\mathsf{L} := \mathsf{S4} + \{\mathsf{T}(\varphi) \mid \mathsf{L} \vdash \varphi\}$, the least modal companion of L.

Theorem (Maksimova and Rybakov 1974)

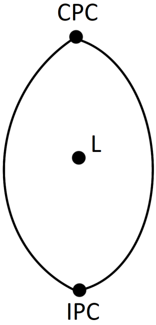
Every superintuitionistic logic L *has a greatest modal companion* $\sigma\mathsf{L}$*.*

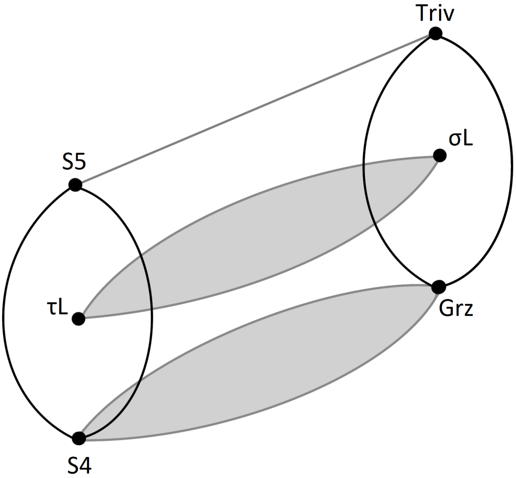
Theorem (Blok-Esakia 1976)

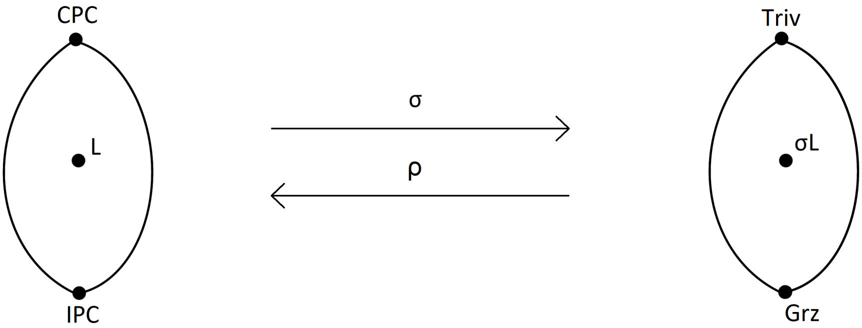
σ is an isomorphism between the lattice of superintuitionistic logics and the lattice of extensions of Grz*, whose inverse is ρ.*

Corollary

$\sigma\mathsf{L} = \mathsf{Grz} + \{\mathsf{T}(\varphi) \mid \mathsf{L} \vdash \varphi\}$.


CPC
L
IPC


Triv
σL
S5
τL
Grz
S4


CPC
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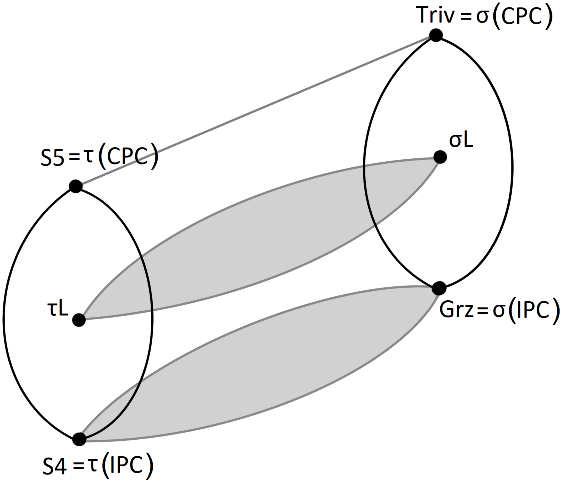
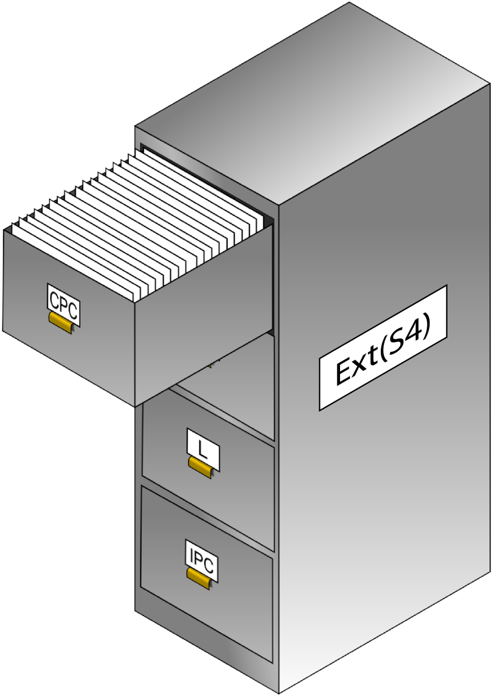

Triv = σ(CPC)
σL
S5 = τ(CPC)
τL
Grz = σ(IPC)
S4 = τ(IPC)


CPC
L
IPC
Ext(S4)

The algebraic proof of the Blok-Esakia Theorem

Definition

A Heyting algebra H is a bounded distributive lattice equipped with a binary operation $\to$ such that for every $a, b, c \in H$:

$$a \wedge b \leq c \iff a \leq b \to c.$$

Theorem (algebraic semantics for IPC)

$\mathsf{IPC} \vdash \varphi$ *iff* $H \vDash \varphi$ *for every Heyting algebra* H.

Definition

An S4-algebra B is a boolean algebra equipped with a unary operator $\Box$ such that such that for every $a, b \in B$:

$$\Box 1 = 1, \qquad \Box(a \wedge b) = \Box a \wedge \Box b, \qquad \Box a \leq a, \qquad \Box a = \Box\Box a.$$

Theorem (algebraic semantics for S4)

$\mathsf{S4} \vdash \varphi$ *iff* $B \vDash \varphi$ *for every S4-algebra* B.

Definition

- If B is an S4-algebra, then $\mathcal{O}(B) := \{b \in B \mid \Box b = b\}$ is a Heyting algebra with $a \to b := \Box(\neg a \vee b)$.
- If H is a Heyting algebra, then the free boolean extension $\mathcal{B}(H)$ of H with the operator

$$\Box\left(\bigwedge_{1}^{n}(\neg a_i \vee b_i)\right) := \bigwedge_{1}^{n}(a_i \to b_i)$$

  is an S4-algebra. In fact, a Grz-algebra.

Theorem

- *If B is an S4-algebra, then $\mathcal{O}(B) \models \varphi$ iff $B \models T(\varphi)$.*
- *If H is an Heyting algebra, then $\mathcal{O}\mathcal{B}(H) \cong H$.*
- *If B is an S4-algebra, then $\mathcal{B}\mathcal{O}(B)$ embeds into B.*

- The category of **finite** Heyting algebras is dually equivalent to the category of finite posets and p-morphisms.
- The category of **finite** S4-algebras is dually equivalent to the category of finite quasi-orders and p-morphisms.

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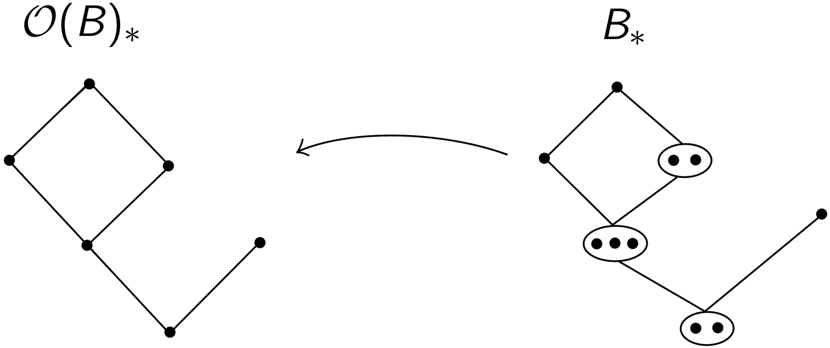

$\mathcal{O}$ corresponds to taking the skeleton of a quasi-order.

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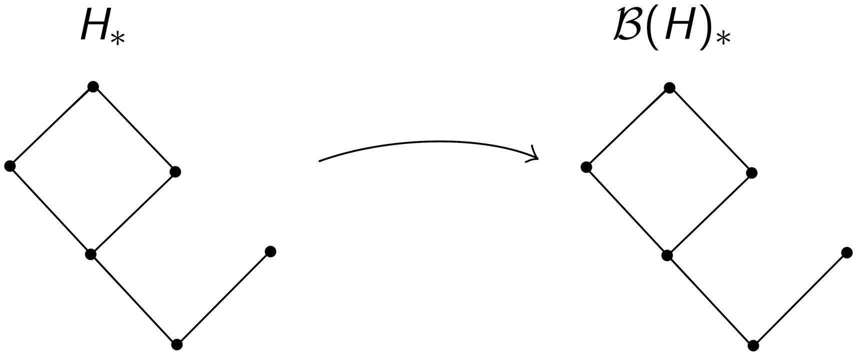
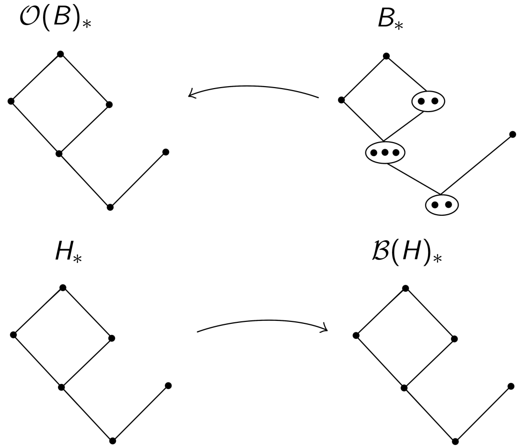


$\mathcal{B}$ corresponds to thinking of a poset as a quasi-order.

- The category of Heyting algebras is dually equivalent to the category of Esakia spaces and continuous p-morphisms.
- The category of S4-algebras is dually equivalent to the category of S4-spaces and continuous p-morphisms.



These operations extend to Esakia spaces and S4-spaces.

$$\text{Superintuitionistic logics} \longleftrightarrow \text{Varieties of Heyting algebras}$$

$$\text{Extensions of S4} \longleftrightarrow \text{Varieties of S4-algebras}$$

If $\mathbb{K}$ is a class of S4-algebras, let $\mathcal{O}(\mathbb{K}) := \{\mathcal{O}(B) \mid B \in \mathbb{K}\}$.

Theorem

- *$\mathcal{O}$ commutes with* H, S, *and* P.
- *If $\mathbb{V}$ is a variety of* S4*-algebras corresponding to an extension* M *of* S4*, then $\mathcal{O}(\mathbb{V})$ is a variety of Heyting algebras and corresponds to* ρM.

If $\mathbb{K}$ is a class of Heyting algebras, let $\mathcal{B}(\mathbb{K}) := \{\mathcal{B}(H) \mid H \in \mathbb{K}\}$.

Proposition

$\mathcal{B}$ commutes with H and S, but it does not commute with infinite products.

Define $\mathcal{B}^*(\mathbb{K}) := \mathsf{HSP}(\mathcal{B}(\mathbb{K}))$. From Maksimova and Rybakov (1974) it follows that:

Theorem

Let $\mathbb{V}$ be a variety of Heyting algebras.

- $\mathcal{B}^*(\mathbb{V})$ *is a variety of* Grz*-algebras.*
- $\mathcal{O}\mathcal{B}^*(\mathbb{V}) = \mathbb{V}$. ***Consequence****: every* L *has a modal companion; i.e., ρ is onto. In particular, τ*L *is the least modal companion of* L.
- $\mathcal{B}^*(\mathbb{V})$ *is the smallest variety $\mathbb{W}$ of* S4*-algebras such that $\mathcal{O}(\mathbb{W}) = \mathbb{V}$.* ***Consequence****: every* L *has a largest modal companion σ*L*, which corresponds to $\mathcal{B}^*(\mathbb{V})$ when* L *corresponds to $\mathbb{V}$.*

Theorem (Blok's Lemma 1976)

- *If B is a* Grz*-algebra, then B and $\mathcal{B}\mathcal{O}(B)$ generate the same variety.*
- *If $\mathbb{W}$ is a variety of* Grz*-algebras, then $\mathcal{B}^*\mathcal{O}(\mathbb{W}) = \mathbb{W}$.*

Therefore, $\mathcal{O}$ (restricted to varieties of Grz-algebras) and $\mathcal{B}^*$ are inverses of each other. **Consequence**: The Blok-Esakia Theorem.

What about the predicate setting?

Rasiowa and Sikorski extended the Gödel translation to the predicate setting as follows:

$$\begin{array}{rcl} \mathsf{T}(\forall x\varphi) & = & \Box\forall x \mathsf{T}(\varphi) \\ \mathsf{T}(\exists x\varphi) & = & \exists x \mathsf{T}(\varphi) \end{array}$$

Theorem (Rasiowa-Sikorski 1953)

T *faithfully embeds the intuitionistic predicate calculus* IQC *into the predicate* S4 *logic* QS4, *i.e.*

$$\mathsf{IQC} \vdash \varphi \qquad \textit{iff} \qquad \mathsf{QS4} \vdash T(\varphi)$$

for any formula φ.

Definition

The monadic fragment (or the one-variable fragment) of a predicate logic L is the set of theorems of L in one fixed variable containing only unary predicate symbols.

Example

$$\forall x(P(x) \to \exists x Q(x))$$

$$\forall(p \to \exists q)$$

Therefore, monadic fragments can be treated like propositional modal logics with additional modalities $\forall$, $\exists$.

Definition

- MIPC is the monadic fragment of IQC.
- MS4 is the monadic fragment of QS4.

The predicate Gödel translation faithfully embeds MIPC into MS4.

$$\begin{aligned} \mathsf{T}(\forall\varphi) &= \Box\forall\mathsf{T}(\varphi) \\ \mathsf{T}(\exists\varphi) &= \exists\mathsf{T}(\varphi) \end{aligned}$$

Let M be an extension of MS4 and L an extension of MIPC.

The intuitionistic fragment of M and modal companions of L are defined similarly to the propositional case.

Definition

- $\rho\mathsf{M} := \{\varphi \mid \mathsf{M} \vdash \mathsf{T}(\varphi)\}$, the intuitionistic fragment of M.
- $\tau\mathsf{L} := \mathsf{MS4} + \{\mathsf{T}(\varphi) \mid \mathsf{L} \vdash \varphi\}$.
- $\sigma\mathsf{L} := \mathsf{MGrz} + \{\mathsf{T}(\varphi) \mid \mathsf{L} \vdash \varphi\}$, where $\mathsf{MGrz} = \mathsf{MS4} + \mathsf{grz}$.

What happens in the monadic setting?

- Is $\tau\mathsf{L}$ a modal companion of L?
- Is $\sigma\mathsf{L}$ a modal companion of L? If so, is it the largest?
- Does Blok-Esakia hold; i.e., is $\sigma\colon \mathsf{Ext}(\mathsf{MIPC}) \to \mathsf{Ext}(\mathsf{MGrz})$ an isomorphism?

Definition

A monadic Heyting algebra H is a Heyting algebra equipped with two unary operators $\forall, \exists$ satisfying for every $a, b \in H$:

$$\forall(a \wedge b) = \forall a \wedge \forall b \qquad \exists(a \vee b) = \exists a \vee \exists b$$
$$\forall 1 = 1 \qquad \exists 0 = 0$$
$$\forall a \leq a \qquad a \leq \exists a$$
$$\forall\exists a = \exists a \qquad \exists\forall a = \forall a$$
$$\exists(\exists a \wedge b) = \exists a \wedge \exists b$$

A monadic S4-algebra (or MS4-algebra) is an S4-algebra equipped with a unary operator $\forall$ satisfying for any $a, b \in B$:

$$\forall(a \wedge b) = \forall a \wedge \forall b \qquad \forall 1 = 1$$
$$\forall a \leq a \qquad a \leq \forall\neg\forall\neg a$$
$$\Box\forall a \leq \forall\Box a$$

Theorem (Algebraic semantics)

- MIPC $\vdash \varphi$ *iff* $H \vDash \varphi$ *for every monadic Heyting algebra* H.
- MS4 $\vdash \varphi$ *iff* $B \vDash \varphi$ *for every* MS4*-algebra* B.

Extensions of MIPC $\longleftrightarrow$ *Varieties of monadic Heyting algebras.*

Extensions of MS4 $\longleftrightarrow$ *Varieties of* MS4*-algebras.*

Definition

- If B is an MS4-algebra, then $(\mathcal{O}(B), \Box\forall, \exists)$ is a monadic Heyting algebra.
- (Fischer Servi 1977) If H is a **finite** monadic Heyting algebra, then the free boolean extension $\mathcal{B}(H)$ can be equipped with a structure of MS4-algebra. It is always a MGrz-algebra.

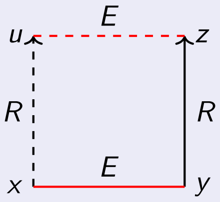
Theorem (Fischer Servi 1977)

If B *is an* MS4*-algebra, then* $\mathcal{O}(B) \vDash \varphi$ *iff* $B \vDash T(\varphi)$.

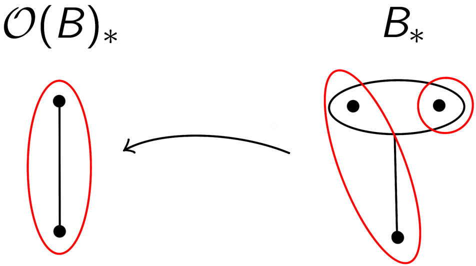
- The category of **finite** monadic Heyting algebras is dually equivalent to the category of finite MIPC-frames.
- The category of **finite** MS4-algebras is dually equivalent to the category of finite MS4-frames.

Definition

An MIPC-frame (MS4-frame) is a poset (quasi-order) (X, R) equipped with an additional equivalence relation E such that:
xEy and yRz imply there is $u \in X$ s.t. xRu and uEz.

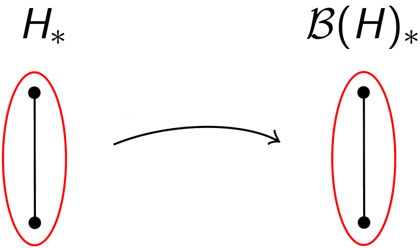


- The category of **finite** monadic Heyting algebras is dually equivalent to the category of finite MIPC-frames.
- The category of **finite** MS4-algebras is dually equivalent to the category of finite MS4-frames.



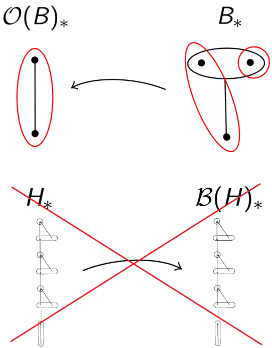
$\mathcal{O}$ corresponds to taking the skeleton of an MS4-frame.

- The category of **finite** monadic Heyting algebras is dually equivalent to the category of finite MIPC-frames.
- The category of **finite** MS4-algebras is dually equivalent to the category of finite MS4-frames.



$\mathcal{B}$ corresponds to thinking of a finite MIPC-frame as an MS4-frame.

- The category of monadic Heyting algebras is dually equivalent to the category of descriptive MIPC-frames.
- The category of MS4-algebras is dually equivalent to the category of descriptive MS4-frames.



Problem

An infinite descriptive MIPC-frame is not always a descriptive MS4-frame.

If $\mathbb{K}$ is a class of MS4-algebras, let $\mathcal{O}(\mathbb{K}) := \{\mathcal{O}(B) \mid B \in \mathbb{K}\}$.

Theorem (Bezhanishvili, C.)

- *$\mathcal{O}$ commutes with* H *and* P*.*
- *$\mathcal{O}\mathsf{S}(\mathbb{K}) \subseteq \mathsf{S}\mathcal{O}(\mathbb{K})$, the other inclusion is not true in general.*
- *If $\mathbb{V}$ is a variety of* MS4*-algebras, then $\mathsf{S}\mathcal{O}(\mathbb{V})$ is the variety generated by $\mathcal{O}(\mathbb{V})$.*

Problem

If $\mathbb{V}$ is a variety of MS4-algebras, then $\mathcal{O}(\mathbb{V})$ is not necessarily a variety.

Theorem (Bezhanishvili, C.)

Let $\mathbb{V}$ be a variety of MS4*-algebras corresponding to an extension* M *of* MS4*. Then $\mathsf{S}\mathcal{O}(\mathbb{V})$ is the variety of monadic Heyting algebras corresponding to ρ*M*.*

Theorem (Bezhanishvili, C.)

- $S\mathcal{O}$ *preserves joins of varieties.*
- $S\mathcal{O}$ *does not preserve binary intersections of varieties.*
- $S\mathcal{O}$ *is not one-to-one on varieties of* MGrz*-algebras.*

	Propositional Setting	Monadic Setting
ρ	preserves arbitrary $\bigwedge$ and $\bigvee$ $\rho\colon$ Ext(Grz) $\to$ Ext(IPC) iso	preserves arbitrary $\bigwedge$, but not binary $\vee$ $\rho\colon$ Ext(MGrz) $\to$ Ext(MIPC) is not 1-1
τ	preserves arbitrary $\bigwedge$ and $\bigvee$ $\tau\colon$ Ext(IPC) $\to$ Ext(S4) 1-1	preserves binary $\wedge$ and arbitrary $\bigvee$???
σ	preserves arbitrary $\bigwedge$ and $\bigvee$ $\sigma\colon$ Ext(IPC) $\to$ Ext(Grz) iso	preserves binary $\wedge$ and arbitrary $\bigvee$???

Failure of the monadic Blok-Esakia Theorem (Bezhanishvili, C.)

$\sigma\colon$ Ext(MIPC) $\to$ Ext(MGrz) is not onto. In particular, it is not an isomorphism.

Sketch of the proof:
σ is left adjoint to $\rho\colon$ Ext(MGrz) $\to$ Ext(MIPC), which we have seen is not one-to-one. Therefore, σ cannot be onto.

Three equivalent open problems

- Does every extension of MIPC have a modal companion?
- Is ρ onto?
- Is τ one-to-one?

Proposition

- If L has a modal companion, then the least such is τL.
- If L is Kripke complete, then it has a modal companion.

Does Esakia's Theorem generalize to MIPC?

- Is MGrz a modal companion of MIPC? ✓
- Is MGrz the largest modal companion of MIPC?
- Is there a largest modal companion of MIPC?

Theorem (Bull 1965, Ono 1977, Fischer Servi 1978)

MIPC *has the finite model property.*

Theorem (Esakia 1988)

MGrz *is a modal companion of* MIPC.

While

$$\mathsf{IQC} \vdash \neg\neg\forall x P(x) \to \forall x \neg\neg P(x),$$

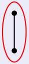
the Kuroda formula $\forall x \neg\neg P(x) \to \neg\neg\forall x P(x)$ is not a theorem of IQC.

Definition

Let Kur $:=$ MIPC $+ \forall\neg\neg p \to \neg\neg\forall p$ be the monadic Kuroda logic.

Kur is a proper extension of MIPC.

Theorem (Esakia-Bezhanishvili 1998)

Kur *is the splitting logic axiomatized by the Jankov formula* $\mathcal{J}($  $)$.

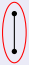
Definition

$\mathsf{GKur} := \mathsf{MS4} + \Box\forall\Diamond\Box p \to \Diamond\Box\forall p.$

$\mathsf{LKur} := \mathsf{MS4} + \Box\forall\Diamond\Box p \to \Diamond\forall p.$

We call GKur the global Kuroda logic and LKur the local Kuroda logic.

Theorem (Bezhanishvili, C.)

- $\mathsf{GKur} = \tau\mathsf{Kur}$ *and is the least modal companion of* Kur.
- LKur *is the splitting logic axiomatized by* $\mathcal{J}($  $)$.

Theorem (Bezhanishvili, C.)

- $\mathsf{LKur} \subsetneq \mathsf{GKur}$.
- LKur *is a modal companion of* MIPC.
- $\mathsf{LKur} \vee \mathsf{MGrz} = \mathsf{GKur} \vee \mathsf{MGrz}$.

Failure of Esakia's Theorem for MIPC (Bezhanishvili, C.)

There is no greatest modal companion of MIPC.

Sketch of the proof:

- LKur and MGrz are both modal companions of MIPC.
- LKur $\vee$ MGrz is not a modal companion of MIPC because

$$\mathsf{GKur} \subseteq \mathsf{GKur} \vee \mathsf{MGrz} = \mathsf{LKur} \vee \mathsf{MGrz}.$$

- There cannot exists a largest modal companion of MIPC because it would contain LKur $\vee$ MGrz, which is not a modal companion of MIPC.

Open problems

By Zorn's Lemma there are maximal modal companions of MIPC.

- How many are there?
- Is MGrz maximal?

THANK YOU!