Modal companions of monadic intuitionistic logic

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Intuitionistic logic is the logic of constructive mathematics and has its origins in Brouwer's criticism of the use of the principle of the excluded middle. It is obtained by weakening the principles of classical logic via the rejection of the law of excluded middle ($p \vee \neg p$). We denote by IPC the intuitionistic propositional calculus.

Modal logic, on the other hand, increases the expressiveness of classical logic by enriching the language with modalities. In this talk, an important role is played by the propositional modal logic S4 introduced by Lewis. This modal system is obtained by adding to the classical propositional calculus a unary modal operator $\Box$ subject to certain axioms.

Quite surprisingly, the Gödel (or Gödel-McKinsey-Tarski) translation allows us to think of IPC as a fragment of S4.

In 1933 Gödel proposed a translation of IPC into S4.

The Gödel translation

$$
\begin{aligned}
\mathsf{T}(\bot) &= \bot \\
\mathsf{T}(p) &= \Box p \\
\mathsf{T}(\varphi \wedge \psi) &= \mathsf{T}(\varphi) \wedge \mathsf{T}(\psi) \\
\mathsf{T}(\varphi \vee \psi) &= \mathsf{T}(\varphi) \vee \mathsf{T}(\psi) \\
\mathsf{T}(\varphi \to \psi) &= \Box(\neg \mathsf{T}(\varphi) \vee \mathsf{T}(\psi))
\end{aligned}
$$

Gödel observed that if IPC $\vdash \varphi$, then S4 $\vdash T(\varphi)$, and conjectured that also the converse holds. This conjecture was eventually established by McKinsey and Tarski.

Theorem (McKinsey-Tarski 1948)

T *embeds* IPC *faithfully into* S4, *i.e.*

$$\mathrm{IPC} \vdash \varphi \qquad \textit{iff} \qquad \mathrm{S4} \vdash T(\varphi)$$

for any formula φ.

Dummett and Lemmon started to investigate the Gödel translation on extensions of IPC.

Definition

Let L be an extension of IPC and M a normal extension of S4.
We call L the intuitionistic fragment of M and M a modal companion of L if T embeds L faithfully into M, i.e. if

$$\mathrm{L} \vdash \varphi \qquad \text{iff} \qquad \mathrm{M} \vdash \mathrm{T}(\varphi)$$

for any formula φ.

By McKinsey-Tarski, S4 is a modal companion of IPC. In fact, it is the least modal companion of IPC.

Each consistent extension of IPC has many modal companions.

Theorem (Grzegorczyk 1967)

$\mathrm{Grz} := \mathrm{S4} + \mathrm{grz}$ *is a modal companion of* IPC, *where*

$$\mathrm{grz} := \Box(\Box(p \to \Box p) \to p) \to p$$

Theorem (Esakia 1976)

Grz *is the greatest modal companion of* IPC.

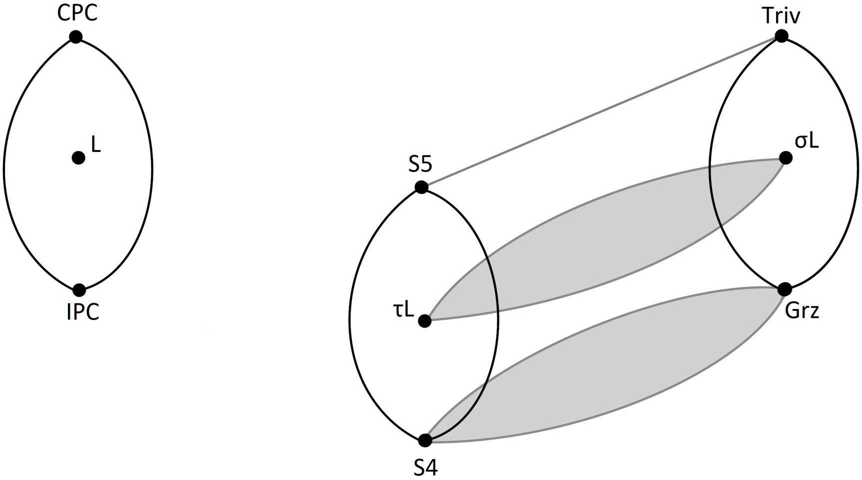
Theorem

Let M *be a normal extension of* S4 *and* L *an extension of* IPC.

- $\rho\mathsf{M} := \{\varphi \mid \mathsf{M} \vdash \mathsf{T}(\varphi)\}$ *is the intuitionistic fragment of* M.
- $\tau\mathsf{L} := \mathsf{S4} + \{\mathsf{T}(\varphi) \mid \mathsf{L} \vdash \varphi\}$ *is the least modal companion of* L.

Theorem (Blok-Esakia 1976)

- $\sigma\mathsf{L} = \mathsf{Grz} + \{\mathsf{T}(\varphi) \mid \mathsf{L} \vdash \varphi\}$ *is the greatest modal companion of* L.
- ρ *restricts to a lattice isomorphism between the lattice of normal extensions of* Grz *and the lattice of extensions of* IPC.
- *The inverse of* ρ *is* σ.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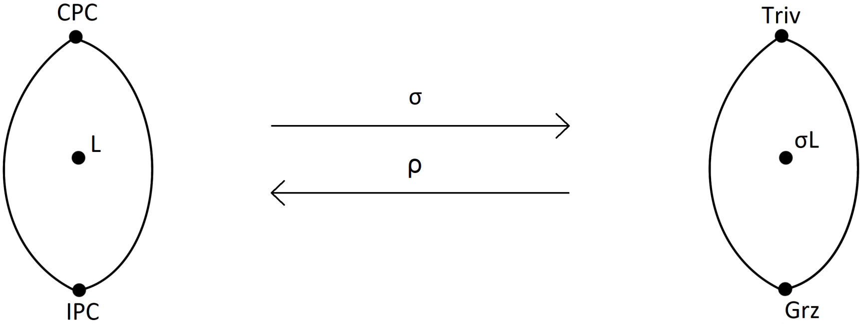
CPC
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Grz

Rasiowa and Sikorski extended the Gödel translation to the predicate setting as follows:

$$\begin{aligned} \mathsf{T}(\forall x \varphi) &= \Box \forall x \mathsf{T}(\varphi) \\ \mathsf{T}(\exists x \varphi) &= \exists x \mathsf{T}(\varphi) \end{aligned}$$

Theorem (Rasiowa-Sikorski 1953)

T *embeds the intuitionistic predicate calculus* IQC *faithfully into the predicate* S4 *logic* QS4*.*

Definition

The monadic fragment of a predicate logic L is given by all the valid formulas of L in one fixed variable that contain only unary relation symbols and no function symbols.

Example

$$\forall x(P(x) \rightarrow \exists x Q(x))$$

Definition

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Example

$$\forall(p \to \exists q)$$

Definition

The monadic fragment of a predicate logic L is given by all the valid formulas of L in one fixed variable that contain only unary relation symbols and no function symbols.

Therefore, monadic fragments can be treated as propositional logics with additional modalities $\forall$, $\exists$. This allows to use semantic tools like algebraic and relational semantics, and powerful duality results.

Definition

- MIPC is the monadic fragment of IQC.
- MS4 is the monadic fragment of QS4.

The predicate Gödel translation restricts to a faithful embedding of MIPC into MS4.

$$\begin{aligned} T(\forall\varphi) &= \Box\forall T(\varphi) \\ T(\exists\varphi) &= \exists T(\varphi) \end{aligned}$$

Theorem

The monadic Grzegorczyk logic $\mathsf{MGrz} := \mathsf{MS4} + \mathsf{grz}$ *is a modal companion of* MIPC.

Theorem (Bezhanishvili-C.)

There is no greatest modal companion of MIPC.

Idea of the proof: We found a modal companion M of MIPC such that the join of M and MGrz is not a modal companion of MIPC.

Let ρ be the map that associates to a normal extension of MS4 its intuitionistic fragment.

Theorem (Bezhanishvili-C.)

The restriction of ρ *to the lattice of normal extensions of* MGrz *is neither a lattice homomorphism nor 1-1.*

Therefore, the natural generalizations of both Esakia's and Blok-Esakia theorems fail in the monadic setting.

Definition

Let

$$\mathsf{M^{+}IPC} := \mathsf{MIPC} + \mathsf{MCas} \quad \text{and} \quad \mathsf{M^{+}Grz} := \mathsf{MGrz} + \mathsf{T(MCas)},$$

where

$$\mathsf{MCas} = \forall((p \to \forall p) \to \forall p) \to \forall p.$$

Theorem (Bezhanishvili-C.)

$\mathsf{M^{+}Grz}$ *is the greatest modal companion of* $\mathsf{M^{+}IPC}$.

Theorem (Bezhanishvili-C.)

ρ *restricts to a lattice isomorphism between the lattice of extensions of* $\mathsf{M^{+}IPC}$ *of finite depth and the lattice of normal extensions of* $\mathsf{M^{+}Grz}$ *of finite depth.*

Conjecture: ρ is also a lattice isomorphism between the lattice of extensions of $\mathsf{M^{+}IPC}$ and the lattice of normal extensions of $\mathsf{M^{+}Grz}$.

THANK YOU!

Axiomatization of MIPC

MIPC is the smallest set of formulas containing

1. all theorems of IPC;
2. the S4-axioms for $\forall$:
   1. $\forall(p \wedge q) \leftrightarrow (\forall p \wedge \forall q)$,
   2. $\forall p \to p$,
   3. $\forall p \to \forall\forall p$;
3. the S5-axioms for $\exists$:
   1. $\exists(p \vee q) \leftrightarrow (\exists p \vee \exists q)$,
   2. $p \to \exists p$,
   3. $\exists\exists p \to \exists p$,
   4. $(\exists p \wedge \exists q) \to \exists(\exists p \wedge q)$;
4. the axioms connecting $\forall$ and $\exists$:
   1. $\exists\forall p \leftrightarrow \forall p$,
   2. $\exists p \leftrightarrow \forall\exists p$;

and closed under the rules of modus ponens, substitution, and necessitation $(\varphi/\forall\varphi)$.

Axiomatization of MS4

MS4 is the smallest set of formulas containing all theorems of the classical propositional calculus CPC, the S4-axioms for $\Box$, the S5-axioms for $\forall$, the left commutativity axiom

$$\Box\forall p \to \forall\Box p,$$

and closed under modus ponens, substitution, $\Box$-necessitation, and $\forall$-necessitation.

Axiomatization of IQC

The intuitionistic predicate logic IQC is the least set of formulas containing all substitution instances of theorems of IPC, the axiom schemes

1. $\forall x A \to A(y/x)$ Universal instantiation
2. $A(y/x) \to \exists x A$
3. $\forall x(A \to B) \to (A \to \forall x B)$ with x not free in A
4. $\forall x(A \to B) \to (\exists x A \to B)$ with x not free in B

and closed under Modus Ponens and Generalization.

Axiomatization of QS4

The modal predicate logic QS4 is the least set of formulas containing all substitution instances of theorems of S4, the axiom schemes

1. $\forall x A \to A(y/x)$ Universal instantiation
2. $\forall x(A \to B) \to (A \to \forall x B)$ with x not free in A

and closed under Modus Ponens, Generalization, and Necessitation.