Ideal and MacNeille completions of subordination algebras

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LATD 2023
Tbilisi, 12 September 2023

With each compact Hausdorff space it is possible to associate various algebraic structures that determine the space up to homeomorphism. This yields dualities for the category KHaus of compact Hausdorff spaces and continuous functions.

In this talk we will consider two dualities from pointfree topology:

- Isbell duality associates with $X \in \mathsf{KHaus}$ the frame of opens of X,
- de Vries duality associates the de Vries algebra of regular opens of X.

We will use subordination algebras to generalize de Vries duality to the category of compact Hausdorff spaces and closed relations between them.

We will then see how to connect these dualities by developing analogues of the ideal and MacNeille completions for subordination algebras.

Definition

A frame is a complete lattice L satisfying the join-infinite distributive law

$$a \wedge \bigvee S = \bigvee \{a \wedge s \mid s \in S\}.$$

A frame L is compact if $\bigvee S = 1$ implies $\bigvee T = 1$ for some finite $T \subseteq S$, and it is regular if $a = \bigvee \{b \in L \mid \neg b \vee a = 1\}$ for each $a \in L$.

Theorem (Isbell duality)

KHaus *is dually equivalent to the category* KRFrm *of compact regular frames and frame homomorphisms.*

- $X \in \mathsf{KHaus}$ is sent to the frame $\mathcal{O}(X)$ of opens of X.
- A continuous function $f \colon X \to Y$ is sent to $\mathcal{O}(f) \colon \mathcal{O}(Y) \to \mathcal{O}(X)$ given by $\mathcal{O}(f)(V) = f^{-1}[V]$.

Definition

A binary relation $R\colon X \to Y$ is closed if $R \subseteq X \times Y$ is a closed subset.

Proposition

Let $R\colon X \to Y$ be a relation between compact Hausdorff spaces. R is closed iff $R[C]$ and $R^{-1}[D]$ are closed for any $C \subseteq X$ and $D \subseteq Y$ closed subsets.

Closed relations are natural generalizations of continuous functions and continuous relations.

Definition

Let $\mathsf{KHaus}^{\mathsf{R}}$ be the category of compact Hausdorff spaces and closed relations.

Let $R_1\colon X \to Y$ and $R_2\colon Y \to Z$ be two relations. We set $x\,(R_2 \circ R_1)\,z$ iff there exists $y \in Y$ such that $x\,R_1\,y$ and $y\,R_2\,z$.

Isbell duality has been generalized to closed relations:

Theorem (Townsend 1996, Jung, Kegelmann, Moshier 2001)

$\mathsf{KHaus}^{\mathsf{R}}$ *is dually equivalent to the category* $\mathsf{KRFrm}^{\mathsf{P}}$ *of compact regular frames and preframe homomorphisms.*

A preframe homomorphism between two frames is a map that preserves directed joins and finite meets.

- $X \in \mathsf{KHaus}$ is sent to the frame $\mathcal{O}(X)$ of opens of X.
- A closed relation $R\colon X \to Y$ is sent to $\mathcal{O}(R)\colon \mathcal{O}(Y) \to \mathcal{O}(X)$ given by $\mathcal{O}(R)(V) = -R^{-1}[-V]$.

Definition

A de Vries algebra is a pair $(B, \prec)$, where B is a complete boolean algebra and $\prec$ a binary relation on B such that

- $1 \prec 1$;
- $a \prec b$ implies $a \leq b$;
- $a \leq b \prec c \leq d$ implies $a \prec d$;
- $a \prec b, c$ implies $a \prec b \wedge c$;
- $a \prec b$ implies $\neg b \prec \neg a$;
- $a \prec b$ implies there is $c \in A$ with $a \prec c \prec b$;
- $a \neq 0$ implies there is $b \neq 0$ with $b \prec a$.

satisfying some properties.

Theorem (de Vries duality)

KHaus *is dually equivalent to the category* DeV *of de Vries algebras and de Vries morphisms.*

Definition

We show that $\mathsf{KHaus}^{\mathsf{R}}$ is equivalent to a category of subordination algebras. We think of compact Hausdorff spaces as quotients of Stone spaces.

Proposition

Compact Hausdorff spaces are, up to homeomorphism, the quotients of Stone spaces over closed equivalence relations.

Definition

Let $\mathsf{StoneE}^{\mathsf{R}}$ be the category defined as follows:

- objects of $\mathsf{StoneE}^{\mathsf{R}}$ are pairs (X, E), where X is a Stone space and E is a closed equivalence relation on X;
- a morphism $R\colon (X_1, E_1) \to (X_2, E_2)$ is a closed relation $R\colon X_1 \to X_2$ that is compatible, i.e. $E_2 \circ R = R = R \circ E_1$.

Theorem

$\mathsf{StoneE}^{\mathsf{R}}$ *and* $\mathsf{KHaus}^{\mathsf{R}}$ *are equivalent.*

The equivalence maps (X, E) to the quotient X/E.

Closed relations between Stone spaces correspond to subordination relations between boolean algebras.

Definition

A subordination $S\colon A \to B$ between boolean algebras is a binary relation such that:

- $0 \mathrel{S} 0$ and $1 \mathrel{S} 1$;
- $a \mathrel{S} c$ and $b \mathrel{S} c$ imply $(a \vee b) \mathrel{S} c$;
- $a \mathrel{S} c$ and $a \mathrel{S} d$ imply $a \mathrel{S} (c \wedge d)$;
- $a \leq b \mathrel{S} c \leq d$ implies $a \mathrel{S} d$.

By Celani (2018) or Jung, Kurz, and Moshier (2019), it follows that Stone duality can be extended to closed relations.

Theorem

The category of boolean algebras and subordination relations is equivalent to the category of Stone spaces and closed relations.

Definition

A subordination $S\colon B \to B$ is an S5-subordination if for all $a, b \in B$:

- $a \; S \; b$ implies $a \leq b$;
- $a \; S \; b$ implies $\neg b \; S \; \neg a$;
- $a \; S \; b$ implies there is $c \in B$ such that $a \; S \; c$ and $c \; S \; b$.

Let $\mathsf{SubS5^S}$ the category defined as follows:

- objects of $\mathsf{SubS5^S}$ are pairs (B, S), where B is a boolean algebra and S is an S5-subordination on B (called S5-subordination algebras);
- a morphism $T\colon (B_1, S_1) \to (B_2, S_2)$ is a subordination $T\colon B_1 \to B_2$ that is compatible, i.e. $T \circ S_1 = T = S_2 \circ T$.

The equivalence of the previous slide can be lifted.

Theorem

$\mathsf{SubS5^S}$ *is equivalent to* $\mathsf{StoneE^R}$, *and hence to* $\mathsf{KHaus^R}$.

Proposition

A de Vries algebra is an S5-subordination algebra $(B, \prec)$ *such that* B *is a complete boolean algebra and*

- $a \neq 0$ *implies there is* $b \neq 0$ *such that* $b \prec a$.

Definition

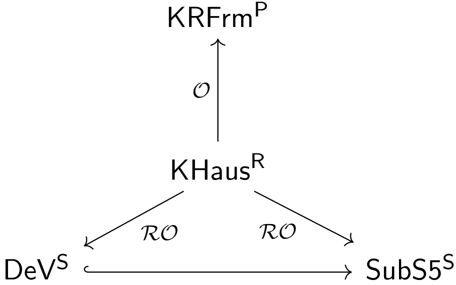
Let $\mathsf{DeV}^{\mathsf{S}}$ be the full subcategory of $\mathsf{SubS5}^{\mathsf{S}}$ consisting of de Vries algebras.

De Vries duality generalizes to closed relations.

Theorem (Abbadini, Bezhanishvili, C.)

$\mathsf{KHaus}^{\mathsf{R}}$ *and* $\mathsf{DeV}^{\mathsf{S}}$ *are equivalent.*

The functors establishing this equivalence behave as the ones of de Vries duality on objects.
A closed relation $R \colon X \to Y$ is mapped to the subordination $\mathcal{RO}(R)$ given by $U \, \mathcal{RO}(R) \, V$ iff $R[\mathrm{cl}(U)] \subseteq V$.


KRFrm^P
𝒪
KHaus^R
ℛ𝒪
ℛ𝒪
DeV^S
SubS5^S

Definition

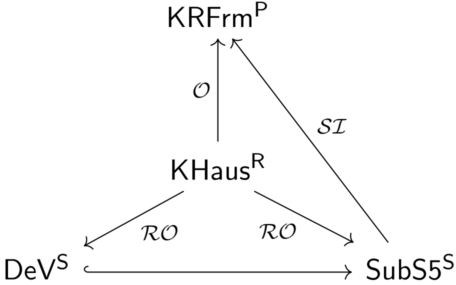
A subordination ideal of an S5-subordination algebra (B, S) is an ideal I of B such that $S^{-1}[I] = I$. The subordination ideals of (B, S) form a compact regular frame, denoted $\mathcal{SI}(B, S)$.

$\mathcal{SI}$ becomes a contravariant functor by mapping a compatible subordination T to the map given by $I \mapsto T^{-1}[I]$.

Theorem (Abbadini, G. Bezhanishvili, C.)

$\mathcal{SI}\colon \mathsf{SubS5^S} \to \mathsf{KRFrm^P}$ *is a dual equivalence.*

$\mathcal{SI}$ is a generalization of the ideal completion. Indeed, if $S = \leq$ on B, then $\mathcal{SI}(B, \leq)$ is exactly the ideal completion of B.


KRFrm^P
O
SI
KHaus^R
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RO
DeV^S
SubS5^S

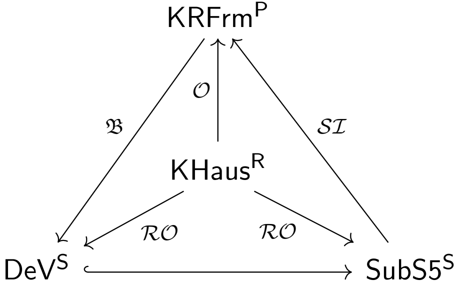
Definition

If L is a compact regular frame, then the set $\mathfrak{B}(L) = \{a \in L \mid \neg\neg a = a\}$ form a boolean algebra, called the booleanization of L. Moreover, $(\mathfrak{B}(L), \prec)$ is a de Vries algebra with $a \prec b$ iff $\neg a \vee b = 1$.

$\mathfrak{B}$ becomes a contravariant functor by mapping a preframe homomorphism $f\colon L_1 \to L_2$ to the compatible subordination relation $\mathfrak{B}(f)\colon (\mathfrak{B}(L_2), \prec_2) \to (\mathfrak{B}(L_1), \prec_1)$ given by $b\, \mathfrak{B}(f)\, a$ iff $b \prec_2 f(a)$.

Theorem (Abbadini, G. Bezhanishvili, C.)

$\mathfrak{B}\colon \mathsf{KRFrm}^{\mathsf{P}} \to \mathsf{DeV}^{\mathsf{S}}$ *is a dual equivalence.*


KRFrm^P
KHaus^R
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We denote by $\mathcal{NI}$ the composition $\mathfrak{B} \circ \mathcal{SI}$.

Corollary

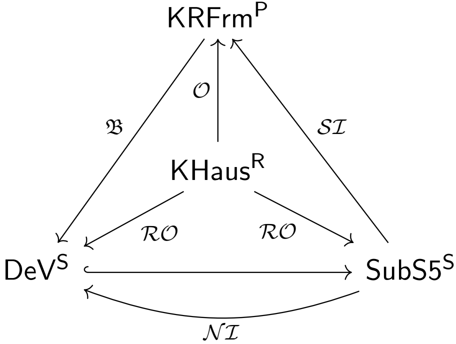
$\mathcal{NI}\colon \mathsf{SubS5^S} \to \mathsf{DeV^S}$ *is an equivalence.*

If (B, S) is an S5-subordination algebra, then $\mathcal{NI}(B, S)$ is a de Vries algebra whose elements are the regular elements of the frame of subordination ideals of (B, S) that we call normal subordination ideals.

Proposition

A subordination ideal I of (B, S) is normal iff $I = S^{-1}[L(S[U(I)])]$, where $L(X)$ and $U(X)$ denote the sets of lower and upper bounds of a subset $X \subseteq B$.

$\mathcal{NI}$ is a generalization of the MacNeille completion. Indeed, if $S = \leq$, then $\mathcal{NI}(B, \leq)$ is the MacNeille completion of B.



$\mathsf{KRFrm}^{\mathsf{P}}$
$\mathcal{O}$
$\mathfrak{B}$
$\mathcal{SI}$
$\mathsf{KHaus}^{\mathsf{R}}$
$\mathcal{RO}$
$\mathcal{RO}$
$\mathsf{DeV}^{\mathsf{S}}$
$\mathsf{SubS5}^{\mathsf{S}}$
$\mathcal{NI}$

Definition

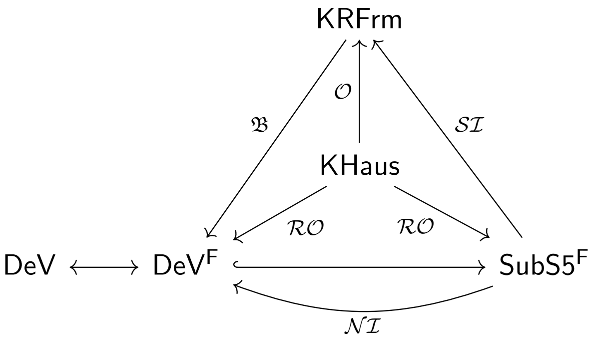
We say that a compatible subordination $T\colon (B_1, S_1) \to (B_2, S_2)$ is functional if

- If $a\ T\ 0$, then $a = 0$.
- If $a\ T\ (b_1 \vee b_2)$, $b_1\ S_2\ b_1'$, and $b_2\ S_2\ b_2'$, then there are $a_1, a_2 \in B_1$ such that $a\ S_1\ (a_1 \vee a_2)$, $a_1\ T\ b_1'$, and $a_2\ T\ b_2'$.

Let $\mathsf{SubS5}^{\mathsf{F}}$ ($\mathsf{DeV}^{\mathsf{F}}$) be the categories of S5-subordination algebras (de Vries algebras) and functional compatible subordinations.

Theorem (Abbadini, G. Bezhanishvili, C. 2022)

- $\mathcal{RO}$ *restricts to equivalences between* KHaus, $\mathsf{SubS5^F}$, *and* $\mathsf{DeV^F}$.
- DeV *is dually isomorphic to* $\mathsf{DeV^F}$.
- *The following diagram of equivalences and dual equivalences commutes up to natural isomorphism.*



Definition

A closed relation $R \colon X \to Y$ is called continuous if $R^{-1}[O]$ is open in X for any open subset O of Y.

Let $\mathsf{KHaus}^{\mathsf{C}}$ be the category of compact Hausdorff spaces and continuous relations between them.

Continuous relations are a natural generalization of continuous functions and play an important role in dualities for Heyting algebras and modal algebras.

Theorem (G. Bezhanishvili, Gabelaia, Harding, Jibladze 2019)

$\mathsf{KHaus}^{\mathsf{C}}$ *is dually equivalent to the categories* $\mathsf{DeV}^{\mathsf{C}}$ *and* $\mathsf{KRFrm}^{\mathsf{C}}$ *that generalize* DeV *and* KRFrm*.*

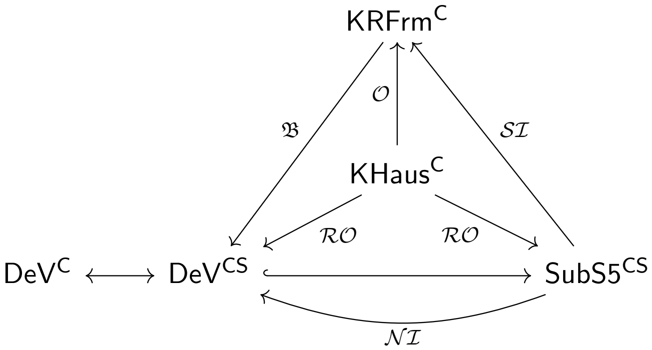
Definition

A compatible subordination $T \colon (B_1, S_1) \to (B_2, S_2)$ is continuous if $S_2[b] \subseteq T[U(T^{-1}[b])])$ for any $b \in B_2$.

Let $\mathsf{SubS5}^{\mathsf{CS}}$ ($\mathsf{DeV}^{\mathsf{CS}}$) be the categories of S5-subordination algebras (de Vries algebras) and continuous compatible subordinations.

Theorem (Abbadini, G. Bezhanishvili, C. 2022)

- *$\mathcal{RO}$ restricts to equivalences between* $\mathsf{KHaus}^{\mathsf{C}}$, $\mathsf{SubS5}^{\mathsf{CS}}$, *and* $\mathsf{DeV}^{\mathsf{CS}}$.
- $\mathsf{DeV}^{\mathsf{C}}$ *is dually isomorphic to* $\mathsf{DeV}^{\mathsf{CS}}$.
- *The following diagram of equivalences and dual equivalences commutes up to natural isomorphism.*



მადლობა

(Thank you)

By Celani (2018) or Jung, Kurz, and Moshier (2019), it follows that Stone duality can be extended to closed relations.

Theorem

The category of boolean algebras and subordination relations is equivalent to the category of Stone spaces and closed relations.

The functors behave as the ones of Stone duality on objects and as follows on morphisms:

- A closed relation $R\colon X \to Y$ is sent to $S_R\colon \mathrm{Clop}(X) \to \mathrm{Clop}(Y)$ defined by $U\, S_R\, V$ iff $R[U] \subseteq V$.
- A subordination relation $S\colon A \to B$ is sent to $R_S\colon \mathrm{Uf}(A) \to \mathrm{Uf}(B)$ defined by $x\, R_S\, y$ iff $S[x] \subseteq y$.