Free algebras and coproducts in varieties of Gödel algebras

Luca Carai, University of Milan

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Free Heyting algebras

Definition

A Heyting algebra H is a distributive lattice equipped with a binary operation $\to$ satisfying

$$a \wedge b \leq c \quad \text{iff} \quad a \leq b \to c$$

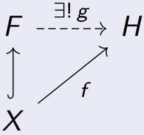
for any $a, b, c \in H$.

Heyting algebras provide the algebraic semantics for the intuitionistic propositional calculus IPC.

Intuitionistic logic is the logic of constructive mathematics and is obtained by weakening the principles of classical logic via the rejection of the law of excluded middle ($p \vee \neg p$).

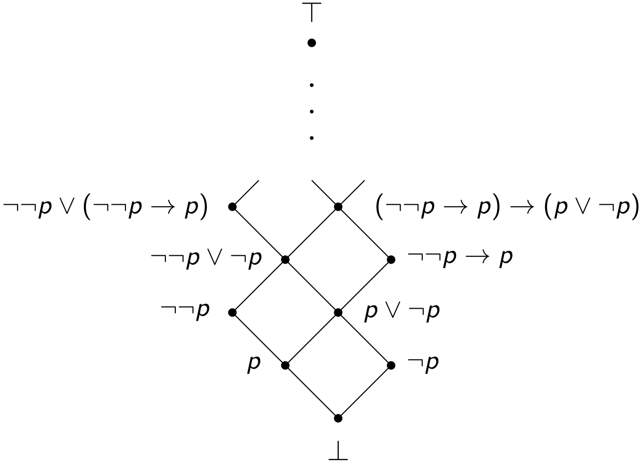
Definition

A Heyting algebra F is said to be free over $X \subseteq F$ if for any Heyting algebra H and function $f\colon X \to H$, there is a unique Heyting algebra homomorphism $g\colon F \to H$ extending f.



- X generates F.
- Two Heyting algebras free over two sets of the same cardinality are isomorphic.
- Let Form(X) be the set of formulas with variables from a set X and define $\varphi \sim \psi$ iff $\vdash_{\mathsf{IPC}} \varphi \leftrightarrow \psi$. Then Form$(X)/{\sim}$ is the Heyting algebra free over X.
- If $X \neq \varnothing$, then the free Heyting algebra over X is infinite.

The free Heyting algebra over 1 generator is also known as the Rieger-Nishimura lattice.



The free Heyting algebra over 2 generators is very complicated.

Definition

An Esakia space is a Stone space X equipped with a partial order $\leq$ such that:

- if $x \in X$, then ${\uparrow}x = \{y \in X \mid x \leq y\}$ is closed;
- if $U \subseteq X$ is clopen (closed and open), then ${\downarrow}U = \{y \in X \mid y \leq x \text{ for some } x \in U\}$ is clopen.

- If X is an Esakia space, then the clopen subsets U of X that are upsets (if $x \in U$ then ${\uparrow}x \subseteq U$) form a Heyting algebra $\mathrm{ClopUp}(X)$.
- To every Heyting algebra H it is possible to associate an Esakia space X such that $H \cong \mathrm{ClopUp}(X)$.
- Morphisms between Esakia spaces are continuous maps that are p-morphisms ($f[{\uparrow}x] = {\uparrow}f(x)$).

Theorem (Esakia duality 1974)

The category of Heyting algebras is dually equivalent to the category of Esakia spaces.

The coloring technique developed by Esakia and Grigolia in the 1970s allows to dually describe the free Heyting algebra F_n over n generators for any $n \in \mathbb{N}$.

The procedure builds a poset X_n as follows:

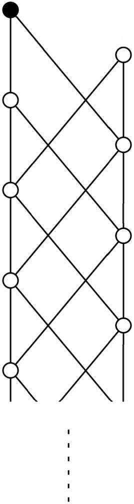
- the layers of X_n are built each one at the time from the top;
- each point constructed is associated with a color, which is an element of $\mathcal{P}(\{1, \dots, n\})$, and the coloring preserves the order;
- the top layer contains 2^n points, one for each color;
- two points of the same color cannot have the same elements as immediate successors;
- if a point has only one immediate successor, then its color should be strictly smaller than the one of the successor.

X_n together with the coloring is known as the n-universal model.

Theorem

- *F_n is isomorphic to a subalgebra of* $\mathrm{Up}(X_n)$.
- *X_n is the dense and open upset of the Esakia dual of F_n consisting of the points of finite depth ($\uparrow x$ is finite).*

X_1 is also known as the Rieger-Nishimura ladder.



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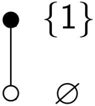
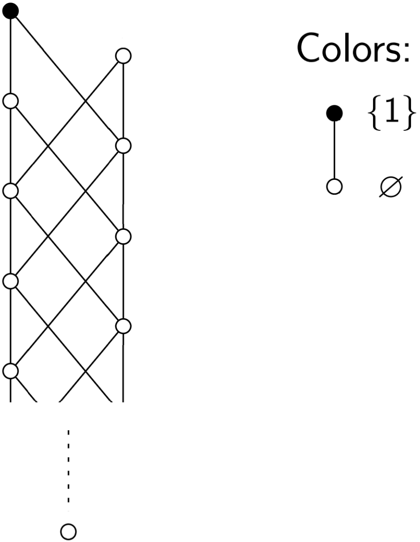


X_1 is also known as the Rieger-Nishimura ladder.

The Esakia dual of F_1 is the following.



However, already the Esakia dual of F_2 is extremely complicated.

Free Gödel algebras

Definition

A Heyting algebra G is called a Gödel algebra if $(a \to b) \vee (b \to a) = 1$ for any $a, b \in G$.

Gödel algebras provide the algebraic semantics for the propositional Gödel-Dummett logic LC, which is obtained by adding the prelinearity axiom $(p \to q) \vee (q \to p)$ to IPC.

We can think of LC as the extension of IPC in which "the truth values are linearly ordered". Indeed, LC is the logic of the class of all finite Heyting chains and the logic of any infinite Heyting chain.

Since LC is the logic of $[0,1]$ as a Heyting chain, it can also be thought of as a fuzzy logic. In fact, LC is a t-norm fuzzy logic with the minimum t-norm.

Definition

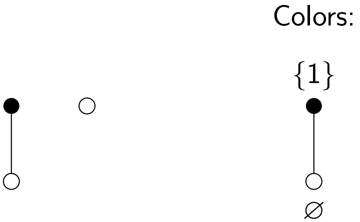
A Gödel algebra F is said to be free over $X \subseteq F$ if for any Gödel algebra G and function $f\colon X \to G$, there is a unique Heyting algebra homomorphism $g\colon F \to G$ extending f.

Proposition

An Esakia space X is dual to a Gödel algebra iff it is a root system; i.e., $\uparrow x$ is a chain for any $x \in X$. We call such spaces Esakia root systems.

We can adapt the construction of the *n*-universal model to LC, by only adding points with a single immediate successor. So, the colors strictly decrease by moving down the layers.

The *n*-universal model for LC is finite and hence coincides with the Esakia dual of the free Gödel algebra on *n* generators.



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$\{1, 2\}$

Proposition

The n-universal model for LC *is isomorphic to the set of all nonempty chains in* $\mathcal{P}(\{1, \ldots, n\})$ *ordered by* $C \leq D$ *iff* D *is an upset of* C.

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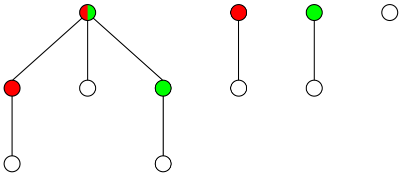

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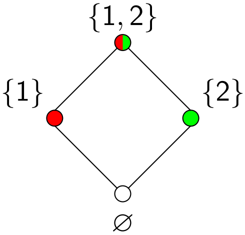

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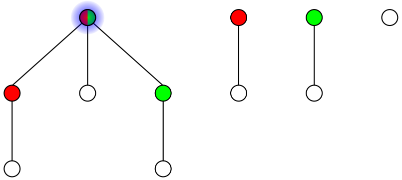

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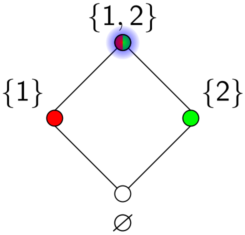

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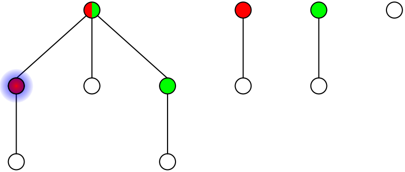

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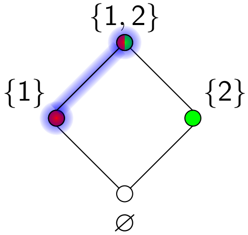

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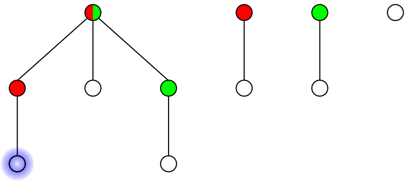

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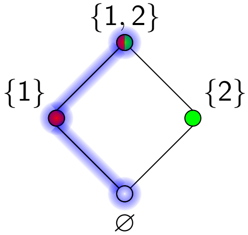

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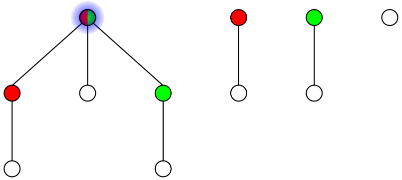

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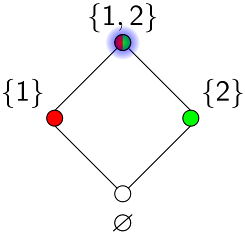

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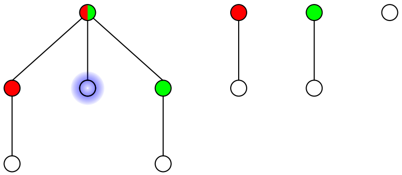

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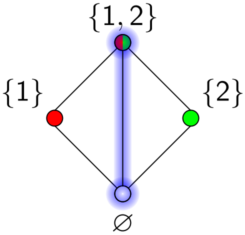

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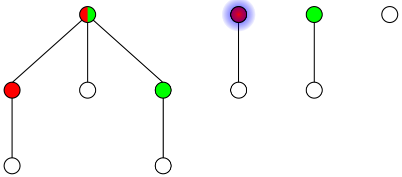

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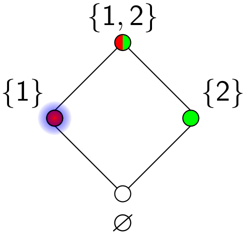

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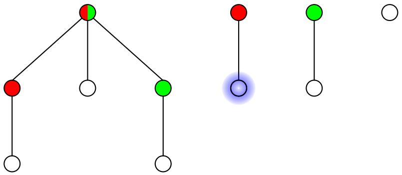

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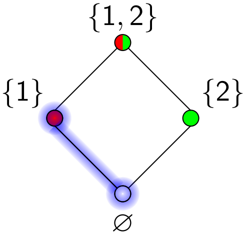

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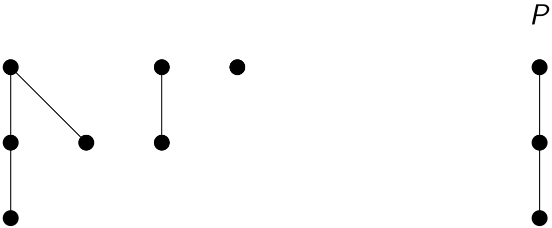
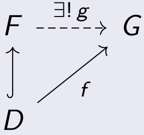


What is the Gödel algebra dual to what you get if you replace $\mathcal{P}(\{1,\ldots,n\})$ with an arbitrary finite poset P?

Definition

A Gödel algebra F is said to be free over a sublattice $D \subseteq F$ if for any Gödel algebra G and lattice homomorphism $f : D \to G$, there is a unique Heyting algebra homomorphism $g : F \to G$ extending f.



Given a distributive lattice D, it is possible to construct a free Gödel algebra over D as a quotient of the free Gödel algebra over the underlying set of D.

Theorem (Aguzzoli, Gerla, and Marra 2008)

Let D be a finite distributive lattice and P a poset such that $D \cong \mathrm{Up}(P)$. The poset of all nonempty chains in P is the Esakia dual of the free Gödel algebra over D.

Every finite distributive lattice is isomorphic to $\mathrm{Up}(P)$ for some poset P.

How to generalize this result to any distributive lattice D?

Definition

A Priestley space is a Stone space X equipped with a partial order $\leq$ such that

- $x \not\leq y$ implies there is U clopen upset such that $x \in U$ and $y \notin U$.

- If X is a Priestley space, then its clopen upsets form a distributive lattice $\mathrm{ClopUp}(X)$.
- To every distributive lattice D it is possible to associate a Priestley space X such that $D \cong \mathrm{ClopUp}(X)$.
- Morphisms between Priestley spaces are continuous maps that preserve the order.

Theorem (Priestley duality 1972)

The category of distributive lattices is dually equivalent to the category of Priestley spaces.

Definition

If X is a Priestley space, we define

$$\mathrm{CC}(X) := \{C \subseteq X \mid C \text{ is a nonempty closed chain}\}.$$

We order $\mathrm{CC}(X)$ by setting $C \leq D$ iff D is an upset of C.

We topologize $\mathrm{CC}(X)$ by the Vietoris topology. A subbasis for the topology on $\mathrm{CC}(X)$ is given by the sets $\Box U$ and $\Diamond U$ for any U clopen of X:

$$\Box U = \{C \in \mathrm{CC}(X) \mid C \subseteq U\},$$
$$\Diamond U = \{C \in \mathrm{CC}(X) \mid C \cap U \neq \varnothing\}.$$

Theorem (C. 2023)

If X is a Priestley space, then $\mathrm{CC}(X)$ is an Esakia root system.

The opens (clopens) of $\mathrm{CC}(X)$ are exactly the (finite) unions of subsets of the form

$$\Box U \cap \Diamond V_1 \cap \cdots \cap \Diamond V_n$$

with $U, V_1, \ldots, V_n$ clopens of X such that $V_1, \ldots, V_n \subseteq U$.

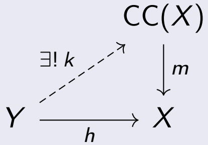
Proposition

Every nonempty closed chain in a Priestley space has a least element.

Let $m\colon \mathrm{CC}(X) \to X$ be the map that sends C to $\min(C)$.

Theorem (C. 2023)

- *m is a continuous order-preserving map.*
- *For any Esakia root system Y and continuous order-preserving map $h\colon Y \to X$ there is a unique continuous p-morphism $k\colon Y \to \mathrm{CC}(X)$ such that $m \circ k = h$.*



- *Let D be a distributive lattice and X its Priestley dual. Then the free Gödel algebra over D is dual to the Esakia space $\mathrm{CC}(X)$.*

Let $\mathbf{2}$ be the 2-element chain with the discrete topology. If S is a set, we consider $\mathbf{2}^S$ with the product topology and the order given by $f \leq g$ iff $f(s) \leq g(s)$ for each $s \in S$.

Proposition

$\mathbf{2}^S$ is a Priestley space dual to the distributive lattice free over S.

The Gödel algebra free over the distributive lattice free over a set S is the Gödel algebra free over S. Therefore,

Theorem (C. 2023)

The Gödel algebra free over a set S is dual to the Esakia space $\mathrm{CC}(\mathbf{2}^S)$.

Since $\mathcal{P}(\{1, \ldots, n\})$ with the discrete topology is isomorphic to $\mathbf{2}^{\{1,\ldots,n\}}$, this generalizes the dual description of finitely generated free Gödel algebras due to the coloring technique.

Ghilardi in 1992 showed that Heyting algebras free over finitely many generators are bi-Heyting algebras.

Definition

Let D be a distributive lattice.

- D is a co-Heyting algebra if its order dual is a Heyting algebra.
- D is a bi-Heyting algebra if it is both a Heyting and a co-Heyting algebra.

Free Gödel algebras over finitely many generators are finite, and so are clearly bi-Heyting algebras. We now see that this is also true with infinitely many generators.

Definition

Let X be a Priestley space.

- X is a co-Esakia space if $(X, \geq)$ is an Esakia space.
- X is a bi-Esakia space if it is both an Esakia and a co-Esakia space.

Co-Esakia spaces are dual to co-Heyting algebras and bi-Esakia spaces are dual to bi-Heyting algebras.

Theorem (C. 2023)

The free Gödel algebra over a distributive lattice D is a bi-Heyting algebra whenever D is a co-Heyting algebra.

Proof (sketch).

- Suppose X is a co-Esakia space. We show that $\mathrm{CC}(X)$ is bi-Esakia.
- Define $\Uparrow(V_1, \dots, V_n) = \uparrow(\cdots\uparrow(\uparrow(\uparrow V_1 \cap V_2) \cap V_3) \cdots \cap V_n)$.
- If $V_1, \dots, V_n$ are clopens, then $\Uparrow(V_1, \dots, V_n)$ is clopen in X.
- $\uparrow(\Box U \cap \Diamond V_1 \cap \cdots \cap \Diamond V_n)$ equals

$$\Box U \cap \bigcup_{I \subseteq \{1,\dots,n\}} \left[\left(\bigcap_{i \in I} \Diamond V_i \right) \cap \bigcup_{\substack{\{j_1,\dots,j_k\} \\ = \{1,\dots,n\} \setminus I}} \Box \Uparrow(V_{j_1}, \dots, V_{j_k}) \right],$$

which is clopen in $\mathrm{CC}(X)$ because X is co-Esakia.

□

Theorem (C. 2023)

Free Gödel algebras are bi-Heyting algebras.

Proof (sketch).

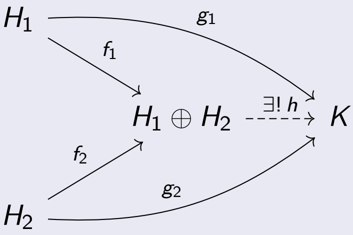
- The Gödel algebra free over a set S is the Gödel algebra free over the distributive lattice free over S.
- Free distributive lattices are co-Heyting algebras (actually bi-Heyting).

□

Coproducts of Gödel algebras

Definition

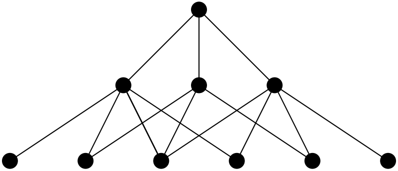
Let $\{H_i\}$ be a family of Heyting algebras. A Heyting algebra $\bigoplus_i H_i$ is called the coproduct of the H_i's if there are Heyting algebra homomorphisms $f_i \colon H_i \to \bigoplus_i H_i$ such that for any Heyting algebra K and Heyting algebra homomorphisms $g_i \colon H_i \to K$ there exists a unique Heyting algebra homomorphism $h \colon \bigoplus_i H_i \to K$ and $h \circ f_i = g_i$ for each i.



Coproducts of Gödel algebras are defined similarly.

Coproducts of Heyting algebras are complicated. In 2006 Grigolia dually described coproducts of two finite Heyting algebras.

If $\mathbf{3}$ is the 3-element Heyting chain, then $\mathbf{3} \oplus \mathbf{3}$ is infinite. The following are the first 3 layers of its Esakia dual.

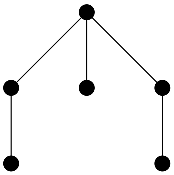


The size of the layers grow exponentially:

- the 4^{th} layer has 72 points
- and the 5^{th} layer has more than 10^{21} points.

Coproducts of Gödel algebras are much simpler. D'Antona and Marra in 2006 dually described the coproduct of two finite Gödel algebras, which is always finite.

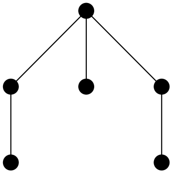
If **3** is the 3-element chain thought of as a Gödel algebra, then $\mathbf{3} \oplus \mathbf{3}$ is finite (it has 22 elements). The following is its Esakia dual.

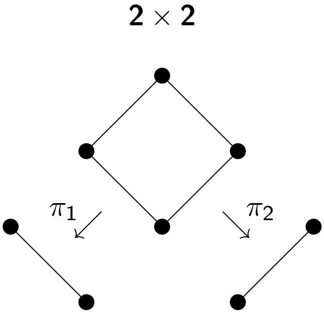


Let $\{X_i\}$ be a family of Esakia root systems. We denote by $\prod_i X_i$ their cartesian product with the product topology and the product order.

Definition

Let $\bigotimes_i X_i$ be the subspace of $\mathrm{CC}(\prod_i X_i)$ given by the closed chains in $\prod_i X_i$ such that $\pi_i[C]$ is a principal upset of X_i for each $i \in I$.





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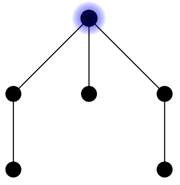

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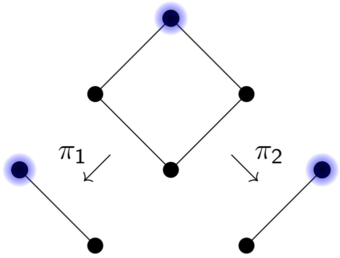

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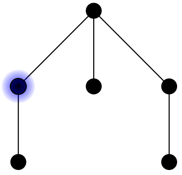

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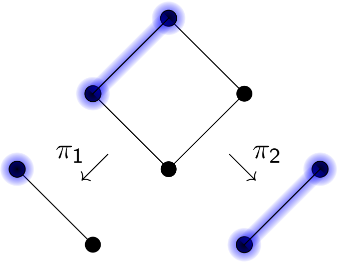

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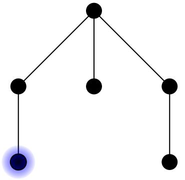
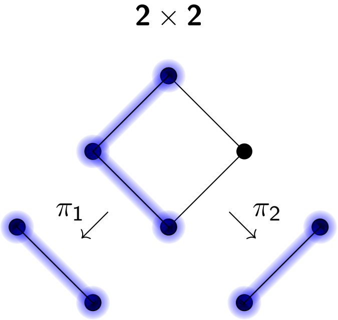
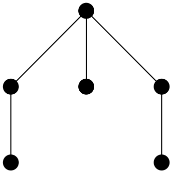




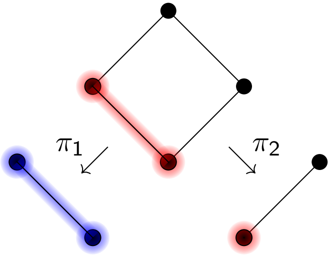
Let $\{X_i\}$ be a family of Esakia root systems. We denote by $\prod_i X_i$ their cartesian product with the product topology and the product order.

Definition

Let $\bigotimes_i X_i$ be the subspace of $CC(\prod_i X_i)$ given by the closed chains in $\prod_i X_i$ such that $\pi_i[C]$ is a principal upset of X_i for each $i \in I$.



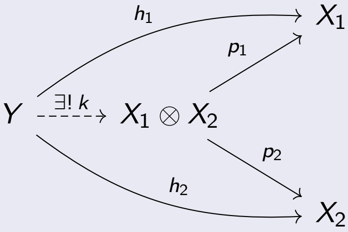
$\mathbf{2} \times \mathbf{2}$



Let $p_i \colon \bigotimes_i X_i \to X_i$ be the map that sends C to $\pi_i(\min C)$.

Theorem (C. 2023)

- $\bigotimes_i X_i$ *is an Esakia root system and each* p_i *is a continuous p-morphism.*
- *For any Esakia root system* Y *and continuous p-morphisms* $h_i \colon Y \to X_i$*, there is a unique continuous p-morphism* $k \colon Y \to \bigotimes_i X_i$ *such that* $p_i \circ k = h_i$ *for each* i*.*



- *Let* $\{G_i\}$ *be a family of Gödel algebras and* $\{X_i\}$ *their dual Esakia root systems. Then* $\bigoplus_i G_i$ *is dual to* $\bigotimes_i X_i$*.*

The case of Gödel algebras of bounded depth

Each extension of LC is of the form $\mathsf{LC}_n := \mathsf{LC} + \mathsf{bd}_n$, where bd_n is the bounded depth n axiom for $n \in \mathbb{N}$.

The algebraic semantics for LC_n is given by the Gödel algebras validating bd_n. We denote their category by GA_n.

Proposition

A Gödel algebra is in GA_n iff their dual Esakia space has depth at most n.

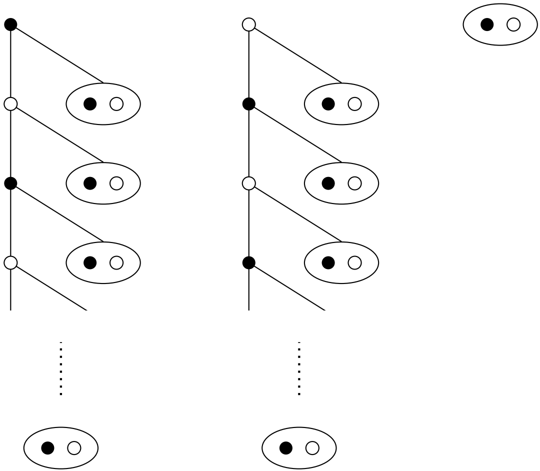
Theorem (C. 2023)

- *Let D be a distributive lattice dual to the Priestley space X. The GA_n-algebra free over D is dual to the subspace of $\mathsf{CC}(X)$ given by the chains of length at most n.*
- *Let $\{G_i\}$ be a family of algebras in GA_n dual to the Esakia root systems $\{X_i\}$. Their coproduct in GA_n is dual to the subspace of $\bigotimes_i X_i$ given by the chains of length at most n.*

Future work

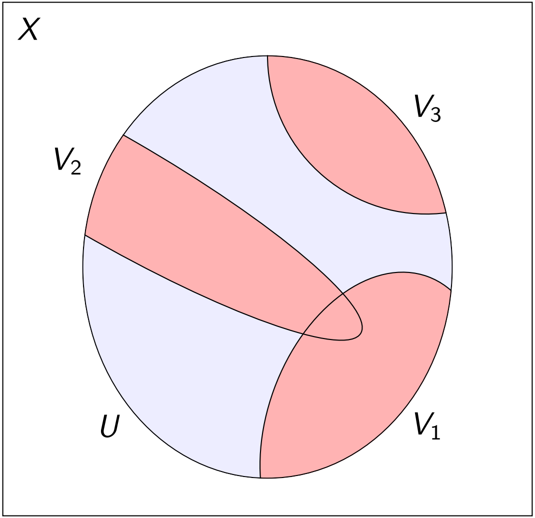
The modal logic S4.3 is the extension of the logic S4 axiomatized by $\Box(\Box p \to q) \vee \Box(\Box q \to p)$. It is the least modal companion of LC.

Esakia and Grigolia in 1975 described the dual of the free S4.3-algebra with 1 generator, which is infinite.

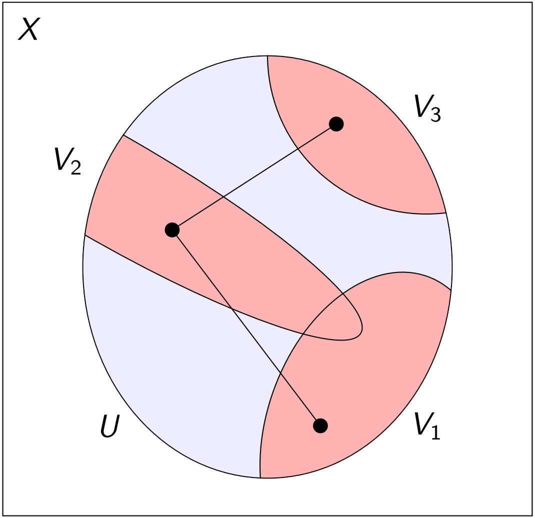


THANK YOU!

$$\Box U \cap \Diamond V_1 \cap \Diamond V_2 \cap \Diamond V_3$$
$$= \{C \in \mathrm{CC}(X) \mid C \subseteq U,\ C \cap V_1 \neq \varnothing,\ C \cap V_2 \neq \varnothing,\ C \cap V_3 \neq \varnothing\}$$



$$\Box U \cap \Diamond V_1 \cap \Diamond V_2 \cap \Diamond V_3$$
$$= \{C \in \mathsf{CC}(X) \mid C \subseteq U,\ C \cap V_1 \neq \varnothing,\ C \cap V_2 \neq \varnothing,\ C \cap V_3 \neq \varnothing\}$$



$$\Box U \cap \Diamond V_1 \cap \Diamond V_2 \cap \Diamond V_3$$
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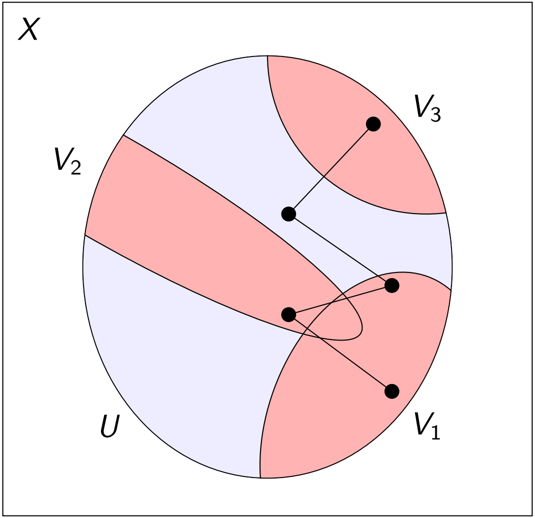

$$\Box U \cap \Diamond V_1 \cap \Diamond V_2 \cap \Diamond V_3$$
$$= \{C \in \mathrm{CC}(X) \mid C \subseteq U,\ C \cap V_1 \neq \varnothing,\ C \cap V_2 \neq \varnothing,\ C \cap V_3 \neq \varnothing\}$$